Furstenburg's x2 x3 theorem.

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In the previous lecture I mentioned two semigroup actions on $\mathbb{R}/\mathbb{Z}$.

(I). $\alpha\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$ by shift.

(II) $\{2^n 3^m \mid n \in \mathbb{Z}^{\geq 0}, m \in \mathbb{Z}^{\geq 0}\} \curvearrowright \mathbb{R}/\mathbb{Z}$ by multiplication

In the first case, we proved that any orbit is equidistributed.

In the second case, Furstenberg proved that

<u>Thm (Furstenberg)</u> $A \subseteq \mathbb{R}/\mathbb{Z}$ closed infinite, invariant under $\times 2$ and $\times 3$ $\Rightarrow A = \mathbb{R}/\mathbb{Z}$.

(Any infinite orbit is dense.)

<u>Remark</u>. Consider the expansion in base $\underline{2}$ of $\alpha :=$ $0.a_0 a_1 a_2 \cdots$. Then $\{2^k \alpha\} = 0.a_k a_{k+1} a_{k+2} \cdots$.

So $\{0, 0.1, 0.11, 0.111, \cdots\}$ is a closed subset of $\mathbb{R}/\mathbb{Z}$ which is invariant under $\underline{\times 2}$.

(Full Bernoulli shift.)

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<u>Conj</u>. Suppose μ is a Borel measure on $\mathbb{R}/\mathbb{Z}$ with no atom. If μ is $\times 2, \times 3$ – invariant and $\times 2, \times 3$ – ergodic, then μ is the Lebesgue measure.

<u>Lemma</u> Suppose Σ is a semigroup of $(\mathbb{Z}^+, \cdot)$ and $\{\log x / \log y \mid x, y \in \Sigma^{>1}\} \not\subseteq \mathbb{Q}$. And $\Sigma = \{s_1 < s_2 < \cdots\}$. Then $\frac{s_{n+1}}{s_n} \longrightarrow 1$.

<u>Pf</u>. $S = \log \Sigma \subseteq \mathbb{R}^{\geq 0}$ is an additive semigroup. So $S-S$ is a subgroup of $\mathbb{R}$. Hence either it is discrete or dense. [If it is NOT discrete, then it has a limit point

$\Rightarrow \exists \{x_n\} \subseteq S-S$ which is a Cauchy sequence, and $x_n \to x \notin S-S$

$\Rightarrow \exists x_n \in S-S$, $x_n \neq 0$ and $x_n \to 0$.

$\Rightarrow \forall \varepsilon > 0$, $y \in \mathbb{R}$, $\exists k \in \mathbb{Z}, n$ s.t. $|y - kx_n| < x_n < \varepsilon$]

So either $S-S = \mathbb{Z}\alpha$ or $\overline{S-S} = \mathbb{R}$.

• <u>The first case</u> implies $S \subseteq \mathbb{Z}\alpha$ which contradicts the assumption.

• <u>The second case</u>: Let $S = \{\overset{\circ}{\overset{\shortparallel}{x_1}} < x_2 < \cdots\}$.

$$\Rightarrow S - S = \bigcup_{n=0}^{\infty} \left(S - (x_1 + \cdots + x_n)\right) \quad \text{I}$$

and $S - (x_1 + \cdots + x_m) \subseteq S - (x_1 + \cdots + x_n)$ if $m \le n$. II

So, if $S - (x_1 + \cdots + x_n)$ has an ε-gap in $(0, \infty)$, then

$$S - (x_1 + \cdots + x_n) - k_{n,\varepsilon} x_1 \text{ has an } \varepsilon\text{-gap in } (0, 2x_1)$$

$$\text{for some } k_{n,\varepsilon} \in \mathbb{Z}^{\ge 0}$$

On the other hand, $S - k_{n,\varepsilon} x_1 \supseteq S \Rightarrow$

$$S - (x_1 + \cdots + x_n) \text{ has an } \varepsilon\text{-gap in } (0, 2x_1).$$

By I, II, $\forall \varepsilon > 0$, $n \gg_{\varepsilon} 1$, $S - (x_1 + \cdots + x_n)$ has no

ε-gap in $[0, 2x_1] \Rightarrow$

$$S - (x_1 + \cdots + x_n) \text{ has no } \varepsilon\text{-gap in } [0, \infty)$$

$\Rightarrow$ "tail" of S has no ε-gap

$\Rightarrow \lim_{n \to \infty} x_{n+1} - x_n = 0$ if $S = \{x_0 < x_1 < \cdots\}$. $\square$

<u>Lemma</u>. $A \subseteq \mathbb{R}/\mathbb{Z}$ closed and Σ-invariant, Σ is

as above. $\exists_{0\neq}\alpha_n \in A$, $\alpha_n \to 0$. Then $A = \mathbb{R}/\mathbb{Z}$.

Pf. $\forall \varepsilon > 0, \forall n \geq N(\varepsilon)$ $\quad s_{n+1} - s_n < \varepsilon s_n$

$\forall m \gg_\varepsilon 1$, $0 < \alpha_m < \dfrac{\varepsilon}{s_{N(\varepsilon)}}$

$\mapsto 0 < s_{N(\varepsilon)} \alpha_m < \varepsilon$

$\Rightarrow \quad s_k \alpha_m < \varepsilon (1+\varepsilon)^{k - N(\varepsilon)}$

So $\quad s_N \alpha_m < s_{N+1} \alpha_m < \cdots < s_{N+k} \alpha_m \leq 1$

is ε-dense in $\mathbb{R}/\mathbb{Z}$. And so we are done. ■

Lemma. $A \subseteq \mathbb{R}/\mathbb{Z}$ closed, $\Sigma_{a,b}$, $\log a, \log b$ $\mathbb{Q}$-linea. independent

$A : \Sigma_{a,b}$- invariant

$\exists \frac{r}{s} \in$ limit set of A

$\Rightarrow A = \mathbb{R}/\mathbb{Z}$.

Pf. $\frac{(ab)^n r}{s} \in$ limit set of $A \Rightarrow$ we can and will assume

$\gcd(abr, s) = 1 \Rightarrow \exists k \in \mathbb{Z}^{>0}$, $a^k \overset{s}{\equiv} 1$ and $b^k \overset{s}{\equiv} 1$

$\Rightarrow x a^k$ and $x b^k$ restricted to $\frac{1}{s}\mathbb{Z}/\mathbb{Z}$ are trivial

$\Rightarrow$ $A - \frac{r}{s}$ is Σ_{a^k, b^k} - invariant, closed and has $\underline{0}$ as a limit point.

$\Rightarrow A - r/s = \mathbb{R}/\mathbb{Z} \Rightarrow A = \mathbb{R}/\mathbb{Z}$.

<u>Main Lemma</u> Suppose A_0 is a minimal set in

$$\{A \subseteq \mathbb{R}/\mathbb{Z} \mid A: \text{closed}, \Sigma_{a,b}\text{-invariant}\}$$

$\Rightarrow A_0$ is a finite set consisting of rational numbers.

<u>Remark</u>. For any A in the above set, there is a minimal set which is a subset of A. To see this, we can use Zorn's lemma: If $A_1 \supseteq A_2 \supseteq \cdots$ is a chain of elements in this set, then $\bigcap_{i=1}^{\infty} A_i$ is a non-empty closed set as A_i's are non-empty compact sets. And clearly $\bigcap_{i=1}^{\infty} A_i$ is Σ-invariant.

<u>Pf</u>. Suppose A_0 is finite $\Rightarrow \forall \bar{x} \in A_0, a \in \Sigma, (a^n - a^m)x \in \mathbb{Z}$

$$\Rightarrow \bar{x} \in \mathbb{Q}/\mathbb{Z}.$$

Suppose A_0 is infinite. Then the set of limit points of A_0 is a closed subset of A_0 which is Σ-invariant. By minimality of A_0, we have that any point of $\underline{A_0}$ is a limit point. By the previous lemma, if A_0 has a rational element, then $A_0 = \mathbb{R}/\mathbb{Z}$ which contradicts minimality of A_0.

So A_0 is a perfect set consisting of irrational points.

Notice that, since A_0 is infinite, 0 is a limit point of $A_0 - A_0$. $A_0 - A_0$ is a closed, Σ-invariant set $\Rightarrow$

$A_0 - A_0 = \mathbb{R}/\mathbb{Z}$.

Let $X_k := \{(x_1 - x_2, x_2 - x_3, \ldots, x_k - x_{k+1}) \mid x_i \in A_0\}$.

<u>Claim</u>. $X_k = \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}$.

<u>Pf of claim</u>. We proceed by induction on $\underline{k}$. The base of induction has been already proved.

By induction hypothesis, for any $r, m_1, \ldots, m_k \in \mathbb{Z}^{\geq 1}$.

$\exists\, x_1, \ldots, x_{k+1} \in A_0$ s.t. $(x_1 - x_2, \ldots, x_k - x_{k+1}) = (\frac{m_1}{r}, \ldots, \frac{m_k}{r})$

Suppose $\gcd(ab, r) = 1 \Rightarrow \exists\, \ell_0 :\ a^{\ell_0} \equiv b^{\ell_0} \equiv 1 \pmod r$

$\Rightarrow\ \times a^{\ell_0}$ and $\times b^{\ell_0}$ restricted to $\frac{1}{r}\mathbb{Z}/\mathbb{Z}$ are identity.

$$\Rightarrow \forall x_{k+2} \in A_0, \quad \left(\frac{m_1}{r}, \dots, \frac{m_k}{r}, a^{l_0 m} b^{l_0 n}(x_{k+1} - x_{k+2})\right)$$
$$\cap$$
$$P_{k+1}.$$

$$\Rightarrow pr_{k+1}\left(P_{k+1} \cap \left\{\frac{m_1}{r}\right\} \times \dots \times \left\{\frac{m_k}{r}\right\} \times \mathbb{R}/\mathbb{Z}\right) \subseteq \mathbb{R}/\mathbb{Z}$$

is a closed, $\Sigma_{a^{l_0}, b^{l_0}}$ - invariant set. And

$x_{k+1} - x_{k+2}$ is in this set for any $x_{k+2} \in A_0$.

Since x_{k+1} is a limit point of A_0, 0 is a limit point of the above set. $\Rightarrow$ this set is $\mathbb{R}/\mathbb{Z}$.

$\Rightarrow \forall r, m_1, \dots, m_k \in \mathbb{Z}^{>1}$, $\gcd(ab, r) = 1$, $\forall y \in \mathbb{R}/\mathbb{Z}$,

$$\left(\frac{m_1}{r}, \dots, \frac{m_k}{r}, y\right) \in P_{k+1}.$$

This set is dense in $(\mathbb{R}/\mathbb{Z})^{k+1}$, which gives us the above claim.

So, for any $k \in \mathbb{Z}^{>1}$, $\exists\, x_1, \dots, x_{k+1} \in A_0$ s.t.

$$x_1 - x_2 = x_2 - x_3 = \cdots = x_k - x_{k+1} = \frac{1}{k}.$$

$\Rightarrow A_0$ is $\frac{1}{k}$-dense in $\mathbb{R}/\mathbb{Z}$ $\Rightarrow$ A_0 is dense in $\mathbb{R}/\mathbb{Z}$

$\Rightarrow A_0 = \mathbb{R}/\mathbb{Z}$, which contradicts minimality. $\blacksquare$

<u>Pf of Furstenberg's theorem.</u> If A is infinite, it has a limit point.

Let A' be the set of limit points of A. Then A' is also closed and $\Sigma_{a,b}$-invariant. Let $A_0 \subseteq A'$ be a minimal $\Sigma_{a,b}$-set. So by the above lemma A_0 consists of rational points. Hence A has a rational point as a limit point.

Therefore by the above lemma $A = \mathbb{R}/\mathbb{Z}$. $\blacksquare$