Counting lattice points and mixing

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Going back Margulis's thesis, we have seen the connection between mixing and counting problems. Here we would like to see a result of Duke-Rudnick-Sarnak using Howe-Moore theorem.

__Theorem__. Let $B_T := \{ g \in SL_n(\mathbb{R}) \mid \|g\| \leq T \}$

$\mathrm{vol}_{\mathbb{Z}}(B_T) := |SL_n(\mathbb{Z}) \cap B_T|$, and

$\mathrm{vol}\,(B_T)$ be the volume of B_T w.r.t. a Haar measure of $SL_n(\mathbb{R})$ s.t. $\mathrm{vol}(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})) = 1$.

Then $\quad \mathrm{vol}_{\mathbb{Z}}(B_T) \sim \mathrm{vol}\,(B_T)$.

__Remark__. D-R-S proved this type of equidistribution of integer points for affine, symmetric, homogeneous varieties: G/H where G and H are reductive, $\tau : G \to G$ is an algebraic involution and $H := \{ g \in G \mid \tau(g) = g \}$. The above example is a particular case: $SL_n \times SL_n / \Delta(SL_n)$.

Later Eskin-Mozes-Shah removed the symmetric assumption, and assume H is maximal and not a parabolic subgroup. Their work is based on Ratner's theorems.

__Pf of theorem__. Let $\widetilde{F}_T(g_1, g_2) := \sum_{\gamma \in SL_n(\mathbb{Z})} \mathbb{1}_{B_T}(g_1^{-1} \gamma g_2)$, and

$$F_T(g_1, g_2) := \frac{1}{\mathrm{vol}\,(B_T)} \widetilde{F}_T(g_1, g_2).$$

<u>Observation</u>. $\tilde{F}_T(g_1\gamma_1, g_2\gamma_2) = \sum_\gamma \mathbb{1}_{B_T}(g_1^{-1}\gamma_1^{-1}\gamma\gamma_2 g_2)$

$$= \sum_\gamma \mathbb{1}_{B_T}(g_1^{-1}\gamma g_2)$$

$$= \tilde{F}_T(g_1, g_2).$$

So we get functions on $G/_\Gamma \times G/_\Gamma$. We would like to show that

$$F_T(I,I) \xrightarrow[T\to\infty]{} 1 .$$

First we prove a weak convergence:

<u>Claim</u>. $F_T([g_1],[g_2]) \xrightarrow[T\to\infty]{} \mathbb{1}_{\Gamma\backslash G\times\Gamma\backslash G}$ weakly in $L^2(\Gamma\backslash G \times \Gamma\backslash G)$.

<u>Pf of claim</u>. Suppose $\phi_1, \phi_2 \in C_c^\infty(\Gamma\backslash G)$

$$\langle F_T([g_1],[g_2]), \phi_1\otimes\phi_2\rangle = \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} \frac{1}{\mathrm{vol}(B_T)} \sum_{\gamma\in\Gamma} \mathbb{1}_{B_T}(g_1^{-1}\gamma g_2)\, \phi_1([g_1])\,\phi_2([g_2])\, d[g_2]\, d[g_1]$$

$$= \frac{1}{\mathrm{vol}(B_T)} \int_{\Gamma\backslash G} \phi_1([g_1]) \int_G \mathbb{1}_{B_T}(g_1^{-1} g)\, \phi_2([g])\, dg \quad d[g_1].$$

$$= \frac{1}{\mathrm{vol}(B_T)} \int_{\mathcal{F}} \phi_1(g_1) \int_{B_T} \phi_2(g_1 g) \quad dg\ dg_1$$

$$= \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} \int_{\mathcal{F}} \phi_1(g_1)\ \rho(g)(\phi_2)(g_1)\ dg_1\ dg$$

$$= \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} \langle \phi_1, \rho(g)\phi_2 \rangle \, dg .$$

By Howe-Moore, $\forall \varepsilon > 0$, if $\|g - I\| \gg_\varepsilon 1$, then $\langle \phi_1, \rho(g)\phi_2 \rangle = \int \phi_1 \int \phi_2 + O_{\phi_1,\phi_2}(\varepsilon)$.

$$\text{So } \langle F_T, \phi_1 \otimes \phi_2 \rangle = \frac{1}{\mathrm{vol}(B_T)} \left(\int_{B_{T_\varepsilon}} \langle \phi_1, \rho(g)\phi_2 \rangle \, dg + \int_{B_T \setminus B_{T_\varepsilon}} \left[\left(\int \phi_1 \int \phi_2 \right) + O(\varepsilon) \right] dg \right)$$

$$= \frac{\int_{B_{T_\varepsilon}} \langle \phi_1, \rho(g)\phi_2 \rangle \, dg}{\mathrm{vol}(B_T)} + \frac{\left(\int \phi_1 \int \phi_2 \right) \left(\mathrm{vol}(B_T) - \mathrm{vol}(B_{T_\varepsilon}) \right)}{\mathrm{vol}(B_T)} + O(\varepsilon) \, \frac{\mathrm{vol}(B_T) - \mathrm{vol}(B_{T_\varepsilon})}{\mathrm{vol}(B_T)}$$

For a fixed ε, let T go to infinity:

$$\langle F_T, \phi_1 \otimes \phi_2 \rangle = \langle \mathbb{1}, \phi_1 \otimes \phi_2 \rangle + O_{\phi_1,\phi_2}(\varepsilon) .$$

$$\Rightarrow F_T \xrightarrow[T \to \infty]{} \mathbb{1}_{\Gamma \backslash G \times \Gamma \backslash G} \quad \text{weakly} .$$

We need the following facts about B_T, in order to get pointwise convergence. (This step is sometimes called <u>well-roundedness</u> of B_T).

① $\mathcal{O}_\varepsilon B_T \mathcal{O}_\varepsilon \subseteq B_{(1+\varepsilon)^2 T}$ where $\mathcal{O}_\varepsilon := \{g \in G \mid \|g^{\pm 1} - I\| \le \varepsilon\}$.

② $\mathrm{vol}\, B_T \sim c\, T^{n^2-n}$

Proof of ② is NOT that easy, but heuristically it should make sense:

$$\left.\begin{array}{l} \mathrm{vol}(\{[x_{ij}] \mid |x_{ij}| \le T\}) = T^{n^2} \\ \det(\{[x_{ij}] \mid |x_{ij}| \le T\}) \subseteq [c' T^n, c T^n] \end{array}\right\} \Rightarrow$$

$$\mathrm{vol}(\{g \in SL_n(\mathbb{R}) \mid \|g\| \le T\}) = c\, T^{n^2-n}.$$

What we need is the following:

$\circledast$ $\exists$ two continuous functions c_1 and c_2 s.t.

$$c_1(\varepsilon) \le \underline{\lim}_{T\to\infty} \frac{\mathrm{vol}(B_{(1-\varepsilon)T})}{\mathrm{vol}(B_T)} \le \overline{\lim}_{T\to\infty} \frac{\mathrm{vol}(B_{(1+\varepsilon)T})}{\mathrm{vol}(B_T)} \le c_2(\varepsilon)$$

and $\lim_{\varepsilon\to 0} c_i(\varepsilon) = 1$.

Clearly $\circledast$ holds because of ②. It might be easier to prove $\circledast$ directly; I leave it as a question for you to try it out:

$\boxed{Q}$ Can one use the following formula

$$\int_{SL_n(\mathbb{R})} f(g)\, dg =$$

$$\int_K \int_{A^+} \int_K \prod_{\varphi\in\Phi^+} \left(\frac{\alpha(a) - \alpha(a)^{-1}}{2}\right) f(k_1 a k_2)\, dk_1\, da\, dk_2$$

to get $\circledast$? Notice that B_T is bi-K-invariant:

$$\mathrm{vol}(B_T) = \int_{(a_1,\dots,a_{n-1})\in\mathbb{R}^{+(n-1)}} \prod_{1\le i<j\le n} \frac{a_i^2 - a_j^2}{2a_i a_j} \frac{da_1 \cdots da_{n-1}}{a_1 \cdots a_{n-1}}$$

$$T \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n$$

where $a_n = (a_1 \cdot \cdots \cdot a_{n-1})^{-1}$.

Suppose $\xi_\varepsilon \in C_c^\infty(G)$ s.t. ① $0 \leq \xi_\varepsilon$

② $\mathrm{supp}(\xi_\varepsilon) \subseteq \mathcal{O}_\varepsilon$

③ $\int \xi_\varepsilon(g)\, dg = 1$.

Let $\hat{\xi}_\varepsilon(\Gamma g) := \sum_{\gamma \in \Gamma} \xi_\varepsilon(\gamma g)$. Then $\mathrm{supp}(\hat{\xi}_\varepsilon) \subseteq \pi(\mathcal{O}_\varepsilon)$ and

$\int_{\Gamma\backslash G} \hat{\xi}_\varepsilon\, d[g] = 1$. (by Weil formula).

$$\langle F_T, \hat{\xi}_\varepsilon \otimes \hat{\xi}_\varepsilon \rangle = \int_{\Gamma\backslash G \times \Gamma\backslash G} F_T([g_1],[g_2]) \sum_{\gamma_1,\gamma_2} \xi_\varepsilon(\gamma_1 g_1)\xi_\varepsilon(\gamma_2 g_2)\, d[g_1]d[g_2]$$

$$= \int_{G\times G} F_T(g_1,g_2)\, \xi_\varepsilon(g_1)\, \xi_\varepsilon(g_2)\, dg_1\, dg_2$$

$$= \int_{\mathcal{O}_\varepsilon \times \mathcal{O}_\varepsilon} F_T(g_1,g_2)\, \xi_\varepsilon(g_1)\, \xi_\varepsilon(g_2)\, dg_1\, dg_2$$

$$F_T(g_1,g_2) = \frac{1}{\mathrm{vol}(B_T)} \sum_{\gamma \in \Gamma} \mathbb{1}_{B_T}(g_1^{-1}\gamma g_2) \leq \frac{\mathrm{vol}(B_{(1+\varepsilon)^2 T})}{\mathrm{vol}(B_T)} F_{(1+\varepsilon)^2 T}(I,I)$$

$$\mathrel{\rotatebox{90}{$\leq$}}$$

$$\frac{\mathrm{vol}(B_{(1+\varepsilon)^{-2} T})}{\mathrm{vol}(B_T)} F_{(1+\varepsilon)^{-2} T}(I,I) \qquad \text{for } g_i \in \mathcal{O}_\varepsilon.$$

$$\Rightarrow \frac{\mathrm{vol}(B_{(1+\varepsilon)^{-2} T})}{\mathrm{vol}(B_T)} F_{(1+\varepsilon)^{-2} T}(I,I) \leq \langle F_T, \hat{\xi}_\varepsilon \otimes \hat{\xi}_\varepsilon \rangle \leq \frac{\mathrm{vol}(B_{(1+\varepsilon)^2 T})}{\mathrm{vol}(B_T)} F_{(1+\varepsilon)^2 T}(I,I)$$

$$\Rightarrow \frac{\mathrm{vol}(B_{(1+\varepsilon)^{-2}T})}{\mathrm{vol}(B_T)} \langle F_{T(1+\varepsilon)^{-2}}, \hat{\xi}_\varepsilon \otimes \hat{\xi}_\varepsilon \rangle \le F_T(I,I) \le \frac{\mathrm{vol}(B_{(1+\varepsilon)^2 T})}{\mathrm{vol}(B_T)} \langle F_{T(1+\varepsilon)^2}, \hat{\xi}_\varepsilon \otimes \hat{\xi}_\varepsilon \rangle$$

Now take $\overline{\lim}_{T\to\infty}$ and $\underline{\lim}_{T\to\infty}$ for a fixed ε and use weak convergence. Then use ⊛ and let $\varepsilon \to 0$. ■