Some Corollaries of Quantitative Non-divergence

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Corollary 1. Suppose $\{u_t\}_{t\in\mathbb{R}^+}$ is a polynomial map. Then for any $x\in\Omega^{(1)}(\mathbb{R}^n)$ $u_t\cdot x \underset{t\to\infty}{\not\to} \infty$.

Remark. This is in contrast to the geodesic flow:

$$a_t = \begin{bmatrix} e^{t/2} & \\ & e^{-t/2}\end{bmatrix} \quad \text{and} \quad \mathbb{Z}^2 \in \Omega^{(1)}(\mathbb{R}^n)$$

$$\Rightarrow \delta(a_t\mathbb{Z}^2) = e^{-|t|/2} \Rightarrow a_t\mathbb{Z}^2 \underset{t\to\infty}{\longrightarrow} \infty.$$

Corollary 2. For any compact subset $C\subseteq\Omega^{(1)}(\mathbb{R}^n)$, and $\varepsilon>0$, there is a compact set C_ε s.t.

$$\forall\, T>0,\ \forall x\in C,\ |\{t\in[0,T] \mid u_t\cdot x\in C_\varepsilon\}| \geq (1-\varepsilon)T.$$

Pf. $\Delta\subseteq x$ primitive $\Rightarrow$ by Minkowski's convex body theorem, $\exists\, v\in\Delta\setminus\{0\}$ s.t. $\|v\|\leq 2\,\mathrm{vol}(B_{\mathbb{R}^k}(1))^{1/k}\, d(\Delta)^{1/2k}$.

$$\Rightarrow \delta(x) \ll d(\Delta)^{1/2k} \quad \Rightarrow 1 \ll_C d(\Delta) \quad \text{for any}$$

primitive subgroup Δ of $x\in C$.

$$\Rightarrow |\{t\in[0,T] \mid \delta(u_t\cdot x)\leq\varepsilon'\}| \ll \varepsilon'^{1/2d_n}\, T$$

Choose ε' small enough so that

$$|\{t\in[0,T] \mid \delta(u_t\cdot x)>\varepsilon'\}| \geq (1-\varepsilon)\,T. \quad \blacksquare$$

Corollary 3. $\forall\varepsilon>0\ \exists\ C_\varepsilon\subseteq\Omega^{(1)}(\mathbb{R}^n)$ a compact set s.t. for any $x\in\Omega^{(1)}(\mathbb{R}^n)$ either

① $\limsup \dfrac{|\{t\in[0,T] \mid u_t\cdot x\in C_\varepsilon\}|}{T} \geq 1-\varepsilon$

or

② x has a primitive subgroup $\Delta = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$

s.t. • $d(\Delta) \leq 1$.

• The $\mathbb{R}$-span of Δ is u_t-invariant.

• $d(u_t\Delta) = d(\Delta)$.

Pf. For any primitive subgroup $\Delta = \bigoplus_{i=1}^{k} \mathbb{Z}v_i$: either

$u_t \cdot (v_1\wedge\cdots\wedge v_k) = v_1\wedge\cdots\wedge v_k \underset{(?)}{\Longrightarrow}$ • $\bigoplus \mathbb{R}v_i$ is u_t-invar.

• $d(u_t\Delta) = d(\Delta)$

or the enteries are non-constant polynomials

$\Rightarrow \|u_t\cdot(v_1\wedge\cdots\wedge v_k)\| \xrightarrow[t\to\infty]{} \infty$.

If ② does NOT hold, $\exists\ T_0>0$ s.t. $\forall\ \Delta \subseteq x$ primitive

$\Rightarrow \max\{d(u_t\Delta) \mid t\in[0,T_0]\} \geq 1$.

(Notice there are only finitely many primitive subgroups Δ of x s.t. $d(\Delta)\leq 1$.) So if $0<\varepsilon' \ll_{\varepsilon} 1$, then

$\frac{1}{T}|\{t\in[0,T] \mid \delta(u_t\cdot x) \leq \varepsilon'\}| \leq \varepsilon$ for $T \geq T_0$

and we get ①. ∎

Remark. Case ② $\Rightarrow \{u_t \cdot x \mid t\in\mathbb{R}\} \subseteq g_0 L(\mathbb{Z}^n)$

where $L = \left\{ \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \;\middle|\; A\in SL_k(\mathbb{R}),\ B\in SL_{n-k}(\mathbb{R}),\ C\in M_{k,n-k}(\mathbb{R}) \right\}$.

In the proof of Furstenberg's $\times 2 \times 3$ theorem we saw the importance of studying properties of a <u>minimal</u> u_t- <u>invariant set</u> for understanding orbit closures.

<u>Corollary</u> 4. Suppose $\{u_t\}$ is a unipotent flow and $Y \subseteq \Omega^{(1)}(\mathbb{R}^n)$ is a closed u_t-invariant set. Then $\exists$ a minimal $\{u_t\}$-invariant subset $Y_0 \subseteq Y$.

<u>Pf.</u> Fix ε (It is NOT important). By the previous corollary, for any x, either a subspace of $x_{\mathbb{Q}}$ is invariant under u_t <u>or</u> $u_t \cdot x$ returns to C_ε more than $1-\varepsilon$ portion of time.

Let k be the max. length of u_t-invar. $\mathbb{Q}$-flags of $x_{\mathbb{Q}}$, among $x\in Y$, and $x_0 \in Y$ s.t.

$$\Delta_1 \lneq \cdots \lneq \Delta_k \lneq x_0$$

① $d(\Delta_i) \leq 1$

② u_t acts trivially on $\wedge^{\operatorname{rank}\Delta_i}(\Delta_i)_{\mathbb{R}}$.

Let $Y_1 := \overline{\{u_t \cdot x_0\}_{t\in\mathbb{R}}} \Rightarrow \forall y\in Y_1$, we have

$u_{t_i} \cdot x_0 \longrightarrow y \Rightarrow \quad u_{t_i} \cdot \Delta_1 \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} u_{t_i} \cdot \Delta_k \underset{\neq}{\leq} u_{t_i} \cdot x_0$

$\Rightarrow (u_{t_i} \Delta_j)_{\mathbb{R}} = (\Delta_j)_{\mathbb{R}}$ and $d(u_{t_i} \Delta_j) = d(\Delta_j)$

$u_{t_i} \cdot x_0$ is convergent $\underset{(?)}{\Rightarrow}$ $u_{t_i} \Delta_j$ converges to $\Delta_j^{(y)}$

a lattice in $(\Delta_j)_{\mathbb{R}}$ and $d(\Delta_j^{(y)}) = d(\Delta_j) \Rightarrow$

$$\Delta_1^{(y)} \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} \Delta_k^{(y)} \underset{\neq}{\leq} y$$

① $d(\Delta_j^{(y)}) \leq 1$.

② u_t acts trivially on $\wedge^{\operatorname{rank} \Delta_j^{(y)}} (\Delta_j^{(y)})_{\mathbb{R}}$.

So by the definition of k, if Δ is comp. with

$$\{\Delta_1^{(y)}, \ldots, \Delta_k^{(y)}\}$$

then either $d(u_t \cdot \Delta) > 1$ or u_t does not act trivially on $\wedge^{\operatorname{rk} \Delta} (\Delta)_{\mathbb{R}}$.

So considering the induced action of u_t on $(\Delta_j)_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}}$ we have that for $0 < \varepsilon \ll 1$ there are compact subsets $C_{j,\varepsilon}$ of

$$\{\overline{\Lambda}_j \subseteq (\Delta_j)_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}} \mid \overline{\Lambda}_j \text{ is a lattice}\}$$

its covolume = covol of Δ_j / Δ_{j-1}

s.t. $\limsup \frac{1}{T} |\{t \in [0, T] \mid u_t \cdot \Delta_j^{(y)} / \Delta_{j-1}^{(y)} \in C_{j,\varepsilon} \ \forall j\}| \geq 1 - \varepsilon$.

s.t. $\limsup \frac{1}{T} \left| \{ t \in [0,T] \mid u_t \cdot \Delta_j^{(y)} / \Delta_{j-1}^{(y)} \in C_{j,\varepsilon}^{-j-1} \ \forall j \} \right| \geq 1-\varepsilon$.

for any $y \in Y_1$.

Notice that, if $u_t \cdot \Delta_j^{(y)} / \Delta_{j-1}^{(y)} \in C_{j,\varepsilon}$ $\underline{\forall j}$, then $\delta(u_t \cdot y)$ is away from zero. [If $v \in u_t \cdot y$ is small and $v \in (\Delta_j)_{\mathbb{R}} \setminus (\Delta_{j-1})_{\mathbb{R}}$, then $v + (\Delta_{j-1})_{\mathbb{R}} \in u_t \Delta_j^{(y)} + (\Delta_{j-1})_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}}$ is small, which is a contradiction.]

$\Rightarrow \exists$ a compact subset C'_ε of $\Omega^{(1)}(\mathbb{R}^n)$ st.

$\forall y \in Y_1$, $\limsup \frac{1}{T} \left| \{ t \in [0,T] \mid u_t \cdot y \in C'_\varepsilon \} \right| \geq 1-\varepsilon$. ⊛

Now let's apply Zorn's lemma in Y_1:

$$\Sigma := \{ Z \subseteq Y_1 \mid Z \text{ closed, non-empty, } u_t\text{-inv.} \}.$$

Let $Z_1 \supseteq Z_2 \supseteq \cdots$ be a chain of elements of Σ.

By ⊛ we have that $Z_i \cap C'_\varepsilon \neq \emptyset$ for any i. So $\{ Z_i \cap C'_\varepsilon \}$ is a chain of non-empty compact sets.

$\Rightarrow \bigcap (Z_i \cap C'_\varepsilon) \neq \emptyset \Rightarrow \bigcap Z_i \in \Sigma$.

Hence by Zorn's lemma Σ has a minimal element. ■

A few comments on the above proof:

- $\forall y \in Y_1$, $\Delta_j^{(y)} = (\Delta_j)_{\mathbb{R}} \cap y \in \Omega((\Delta_j)_{\mathbb{R}})$.
- $(\Delta_j)_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}}$ can be identified with $\Pr_{(\Delta_{j-1})_{\mathbb{R}}^{\perp}} ((\Delta_j)_{\mathbb{R}})$

as a Euclidean space.

• With the above Euclidean structure, we have

$$d(\Delta_j^{(y)}) = d(\Delta_{j-1}^{(y)})\ \mathrm{vol}\left(\Pr_{(\Delta_{j-1})_{\mathbb{R}}^{\perp}}((\Delta_j)_{\mathbb{R}}) \Big/ \Pr_{(\Delta_{j-1})_{\mathbb{R}}^{\perp}}(\Delta_j)\right)^2$$

$$= d(\Delta_{j-1}^{(y)})\ d\left(\Delta_j^{(y)} \big/ \Delta_{j-1}^{(y)}\right)$$

$$\Rightarrow d\left(\Delta_j^{(y)} \big/ \Delta_{j-1}^{(y)}\right) = d\left(\Delta_j / \Delta_{j-1}\right)$$

where we are identifying $\Delta_j^{(y)} / \Delta_{j-1}^{(y)}$ with $\Delta_j + (\Delta_{j-1})_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}}$

and Δ_j / Δ_{j-1} with $\Delta_j + (\Delta_{j-1})_{\mathbb{R}} / (\Delta_{j-1})_{\mathbb{R}}$.

• Clearly the whole issue is the non-compactness of $\Omega^{(1)}(\mathbb{R}^n)$.

• This result was used in Margulis's proof of Oppenheim conjecture.

• What is important is the fact that $\underline{u_t \cdot y}$ returns to the same comp. set for any $y \in Y_1 \subseteq Y$. It is NOT important how often it enters this fixed compact set.

We will discuss the next application later. (after we discuss ergodic actions and some of their properties.)

<u>Corollary 5.</u> Suppose $\{u_t\}$ is a unipotent flow on $\Omega^{(1)}(\mathbb{R}^n)$, and μ is a Radon measure on $\Omega^{(1)}(\mathbb{R}^n)$ which is u_t-invariant and u_t-ergodic. Then μ is a finite measure.

(This result is due to Dani.)

To prove this application we need the following property of Radon measures that are invariant and ergodic w.r.t. a flow:

Suppose $g \in L^1(\mu)$ is non-negative and for a.e. x

$$\int_0^\infty g(u_t x)\, dt = \infty.$$

(This is a recurrence property). Then for any $f \in L^1(\mu)$ and a.e. x we have

$$\lim_{T\to\infty} \frac{\int_0^T f(u_t x)\, dt}{\int_0^T g(u_t x)\, dt} \longrightarrow \frac{\int f(x')\, d\mu(x')}{\int g(x')\, d\mu(x')}$$

[The ratio ergodic theorem.]

Proof of Corollary 5.

Choose $\{a_n\} \subseteq \mathbb{R}^+$ so that $g = \sum_{n=0}^{\infty} a_n \mathbb{1}_{C_{2^{-n}}} \in L^1(\mu)$

where $C_\varepsilon := \{ \Lambda \in \Omega^{(1)}(\mathbb{R}^n) \mid \delta(\Lambda) \geq \varepsilon \}$. Notice that

$\forall\, \varepsilon > 0$, C_ε is a compact set and so $\mu(C_{2^{-n}}) < \infty$ for any n. Hence there is such a sequence $\{a_n\}$.

Claim. For μ-a.e. x, $\int_0^\infty g(u_t x)\, dt = \infty$.

Pf of claim. If for some x

$$\int_0^\infty g(u_t x)\, dt < \infty,$$

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then by Fubini we have, $\forall n$, $\quad \left|\{t\in\mathbb{R}^+ \mid u_t x \in C_{2^{-n}}\}\right| < \infty$. ⊛

On the other hand, by Corollary 2, we have
$\forall x \in \Omega^{(1)}(\mathbb{R}^n)$, if $n \gg_x 1$ and $T \gg_{n,x} 1$, then

$$\left|\{t\in[0,T] \mid u_t x \in C_{2^{-n}}\}\right| \geq 0.9\,T,$$

which contradicts ⊛.

Now applying the Ratio Ergodic Theorem for $f_1 = \mathbb{1}_{C_{\varepsilon_1}}$ and $f_2 = \mathbb{1}_{C_{\varepsilon_2}}$, we have for a.e. x

$$\frac{\int_0^T \mathbb{1}_{C_{\varepsilon_1}}(u_t x)\,dt}{\int_0^T \mathbb{1}_{C_{\varepsilon_2}}(u_t x)\,dt} \xrightarrow[T\to+\infty]{} \frac{\mu(C_{\varepsilon_1})}{\mu(C_{\varepsilon_2})}.$$

Now suppose C is a compact subset of $\Omega^{(1)}(\mathbb{R}^n)$ s.t. $\mu(C)>0$.
By, Corollary 2, $\exists\, \varepsilon_0 = \varepsilon_0(C)$ s.t.

$$\forall x\in C, \quad \left|\{t\in[0,T] \mid u_t x \in C_{\varepsilon_0}\}\right| \geq 0.9\,T$$

$\Rightarrow \forall\, \varepsilon_1, \varepsilon_2 \leq \varepsilon_0$,
$\forall x\in C$
$$0.9 \leq \frac{\int_0^T \mathbb{1}_{C_{\varepsilon_1}}(u_t x)\,dt}{\int_0^T \mathbb{1}_{C_{\varepsilon_2}}(u_t x)\,dt} \leq \frac{1}{0.9}$$

$$\Rightarrow \qquad 0.9 \leq \frac{\mu(C_{\varepsilon_1})}{\mu(C_{\varepsilon_2})} \leq \frac{1}{0.9}$$

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$$\Rightarrow \qquad \mu(\Omega^{(1)}(\mathbb{R}^n)) = \lim_{\varepsilon_1 \to 0} \mu(C_{\varepsilon_1}) \le \frac{1}{0.9}\, \mu(C_{\varepsilon_0}) < \infty.$$

$\blacksquare$