

1 Homework 8.

1. Let k be a field and A be a finitely generated k -algebra. Suppose V is a k -subspace of A such that

- (a) $\dim_k V < \infty$,
- (b) $1 \in V$,
- (c) $A = \bigcup_{i=1}^{\infty} V^{(n)}$ where $V^{(n)} := \{v_1 \cdots v_n \mid v_i \in V\}$.

Let $D_V(n) := \dim_k V^{(n)}$ and

$$\text{GKdim}(A) := \limsup_{n \rightarrow \infty} \frac{\log D_V(n)}{\log n}. \quad (1)$$

- (a) Prove that the value of (1) is independent of choice of V .
- (b) Suppose B/A is a ring extension and A and B are finitely generated k -algebras. Prove that $\text{GKdim}(A) \leq \text{GKdim}(B)$.
- (c) Suppose $\mathfrak{a} \subseteq A$. Prove that $\text{GKdim}(A/\mathfrak{a}) \leq \text{GKdim}(A)$.
- (d) Prove that $\text{GKdim}(A[x]) = \text{GKdim}(A) + 1$.
- (e) Suppose B/A is a ring extension and B is a finitely generated A -module. Prove that $\text{GKdim}(A) = \text{GKdim}(B)$.
- (f) Using the fact that $\dim k[x_1, \dots, x_d] = d$, prove that

$$\text{GKdim}(A) = \dim A.$$

(The importance of GKdimension is on the fact that it works equally well in the non-commutative setting; in fact the above (a)-(e) statements hold for non-commutative algebras as well.)

2. Suppose A is a unital commutative ring. Let $\text{Pic}(A)$ be the class of finitely generated projective A -module up to isomorphism, which are locally free of rank 1; that means

$$\text{Pic}(A) := \{[M] \mid M \text{ finitely generated } A\text{-mod}, \forall \mathfrak{p} \in \text{Spec}(A), M_{\mathfrak{p}} \simeq A_{\mathfrak{p}}\}.$$

- (a) Argue why $[M_1] \cdot [M_2] := [M_1 \otimes_A M_2]$ is a well-defined operation on $\text{Pic}(A)$, and $(\text{Pic}(A), \cdot)$ is a semigroup.

- (b) Argue why $\text{Pic}(A)$ is a commutative monoid and $[A]$ is the identity.
- (c) Suppose $[M] \in \text{Pic}(A)$. Let $M^* := \text{Hom}_A(M, A)$. Prove that

$$[M^*] \in \text{Pic}(A) \quad \text{and} \quad [M^*] \cdot [M] = [A].$$

Deduce that $(\text{Pic}(A), \cdot)$ is a group.

(This is called the Picard group of A .)

3. Suppose D is a Dedekind domain. Recall that this implies $D_{\mathfrak{p}}$ is a PID for every $\mathfrak{p} \in \text{Spec}(D)$ and D is Noetherian. Suppose F is a field of fractions of D .

- (a) Suppose M is a finitely generated D -submodule of F . Prove that, for every $\mathfrak{p} \in \text{Spec}(D)$, $M_{\mathfrak{p}} \simeq D_{\mathfrak{p}}$. Deduce that M is projective and flat.
(Such a D -module is called a fractional ideal, and the set of all fractional ideals is denoted by $\text{Frac}(D)$. The set of cyclic fractional ideals is denoted by $\text{Prin}(D)$. This means $\text{Prin}(D) = \{Da \mid a \in F^\times\}$.)
- (b) Suppose $M_1, M_2 \in \text{Frac}(D)$. Prove that $M_1 M_2 \in \text{Frac}(D)$. Argue why $\text{Frac}(D)$ is a group, D is its identity, and $\text{Prin}(D)$ is a subgroup.
- (c) Prove that if M_1 and M_2 are two fractional ideals, then

$$M_1 \otimes_D M_2 \simeq M_1 M_2.$$

- (d) Suppose $[M] \in \text{Pic}(D)$. Prove that M is isomorphic to a fractional ideal.
- (e) Suppose M_1 and M_2 are two fractional ideals. Prove that $M_1 \simeq M_2$ if and only if there exists $a \in F^\times$ such that $M_2 = aM_1$.
- (f) Prove that $\text{Cl}(D) \simeq \text{Pic}(D)$, where $\text{Cl}(D) := \text{Frac}(D)/\text{Prin}(D)$ is the class group of D .

(Notice that $\text{Cl}(D)$ is trivial if and only if D is a PID.)

(**Hint.** (a) Torsion free finitely generated modules over a PID are free.

(b) Recall that every ideal of D is finitely generated and a finitely generated ideal I is projective if and only if $IM = D$ for some fractional ideal M . For every fractional ideal M , aM is an ideal for some $a \in D \setminus \{0\}$.

(c) Use flatness of M_1 and deduce that

$$0 \rightarrow M_1 \otimes_D M_2 \rightarrow M_1 \otimes_D S^{-1}D \simeq S^{-1}M_1 \simeq F,$$

where $S := D \setminus \{0\}$.

(d) $M \subseteq M_{\mathfrak{p}} \simeq D_{\mathfrak{p}} \subseteq F$.

(e) If $\phi : M_1 \rightarrow M_2$ is an isomorphism, then $S^{-1}\phi : F \rightarrow F$ is an F -linear map. So, $\phi(x) = ax$ for some $a \in F^\times$.

(f) Argue why $M\text{Prin}(D) \mapsto [M]$ is an isomorphism.)

4. Suppose A is a Noetherian ring and $\mathfrak{a}$ is a proper ideal of A . Let $\mathfrak{b} := \bigcap_{i=1}^{\infty} \mathfrak{a}^i$. Prove that $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$.

(**Hint.** Let $\mathfrak{a}\mathfrak{b} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a reduced primary decomposition of $\mathfrak{a}\mathfrak{b}$. Suppose to the contrary that $\mathfrak{a}\mathfrak{b} \neq \mathfrak{b}$. Deduce that $\mathfrak{b} \not\subseteq \mathfrak{q}_i$ for some i . Let $x \in \mathfrak{b} \setminus \mathfrak{q}_i$. Argue why

$$\mathfrak{a} \subseteq (\mathfrak{q}_i : x) \subseteq \mathfrak{p}_i.$$

Deduce that $\mathfrak{a}^m \subseteq \mathfrak{p}_i^m \subseteq \mathfrak{q}_i$ for some $m \in \mathbb{N}$, and say why this is a contradiction.)

5. Suppose A is a local Noetherian ring and $\text{Max}(A) = \{\mathfrak{m}\}$. Prove that $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$.

(**Hint.** Use the previous problem and Nakayama's lemma.)

6. Suppose A is a Noetherian, local, integral domain. Suppose there exists $a \in A$ such that $V(\langle a \rangle) = \{\mathfrak{m}\} = \text{Max}(A)$. Prove that $\dim A \leq 1$.

(**Hint.** Suppose to the contrary that there exists a non-zero prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$. Since $V(\langle a \rangle) = \{\mathfrak{m}\}$, deduce two things (1) $\dim A/\langle a \rangle = 0$, (2) $a \notin \mathfrak{p}$. Conclude that $A/\langle a \rangle$ is Artinian. Show that, for every $n \geq n_0$,

$$\mathfrak{p}^{(n)} + \langle a \rangle = \mathfrak{p}^{(n+1)} + \langle a \rangle,$$

where $\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$. Hence, for every $x \in \mathfrak{p}^{(n)}$, there are $x' \in \mathfrak{p}^{(n+1)}$ and $y \in A$ such that $x = x' + ay$. Using the fact that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$ -primary and $a \notin \mathfrak{p}$, deduce that

$$\mathfrak{p}^{(n)} = \mathfrak{p}^{(n+1)} + a\mathfrak{p}^{(n)}.$$

Argue why

$$\frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}} = \mathfrak{m} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}}, \quad \text{and conclude that} \quad \mathfrak{p}^{(n)} = \mathfrak{p}^{(n+1)}.$$

Say why $\bigcap_{n=1}^{\infty} \mathfrak{p}^n A_{\mathfrak{p}} = \mathfrak{p}^{n_0} A_{\mathfrak{p}}$. Then use the previous problem to obtain a contradiction.)

7. (Krull's Principal Ideal Theorem) Suppose A is Noetherian and $\mathfrak{q}$ is a minimal prime divisor of $\langle a \rangle$. Prove that $\text{ht}(\mathfrak{p}) \leq 1$.

(**Hint.** Suppose to the contrary that there are prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{q}$. Consider $A_{\mathfrak{q}}/\mathfrak{p}_0 A_{\mathfrak{q}}$, and use the previous problem.)

8. (Krull's height theorem) Suppose A is a Noetherian ring and $\mathfrak{p}$ is a minimal prime divisor of $\langle a_1, \dots, a_n \rangle$. Prove that $\text{ht}(\mathfrak{p}) \leq n$.

(**Hint.** Proceed by induction on n . The base case follows from the Krull's PIT. After localization, one can assume that $\mathfrak{p}$ is the unique maximal ideal of A . Suppose $\mathfrak{p}' \subsetneq \mathfrak{p}$ and there is no intermediate prime ideal. Deduce that, there exists a_i such that $a_i \notin \mathfrak{p}'$ – say $a_n \notin \mathfrak{p}'$. Use the assumption that there is no intermediate prime ideal between $\mathfrak{p}'$ and $\mathfrak{p}$, and deduce that $V(\langle a_n \rangle + \mathfrak{p}') = \{\mathfrak{p}\}$. Argue why for every i , there exist $m \in \mathbb{N}$, $r_i \in A$, and $b_i \in \mathfrak{p}'$ such that

$$a_i^m = a_n r_i + b_i.$$

Argue why $V(\langle a_n, b_1, \dots, b_{n-1} \rangle) = \{\mathfrak{p}\}$. Let

$$\bar{\mathfrak{p}} := \mathfrak{p}/\langle b_1, \dots, b_{n-1} \rangle \quad \text{and} \quad \bar{\mathfrak{p}}' := \mathfrak{p}'/\langle b_1, \dots, b_{n-1} \rangle.$$

Use Krull's PIT and show that $\text{ht}(\bar{\mathfrak{p}}) \leq 1$. Deduce that $\text{ht}(\bar{\mathfrak{p}}') = 0$. Argue that $\mathfrak{p}'$ is a minimal prime divisor of $\langle b_1, \dots, b_{n-1} \rangle$. Use the induction hypothesis and deduce that $\text{ht}(\mathfrak{p}') \leq n - 1$. Show that $\text{ht}(\mathfrak{p}) \leq n$.)