

1 Homework 7.

1. Suppose A is an integrally closed domain and F is a field of fractions of A . Let E/F be a finite separable extension, and B the integral closure of A in E . Prove that there exist a basis $e_1, \dots, e_n$ of E over F such that $B \subseteq \sum_{i=1}^n Ae_i$. In particular, deduce that if A is Noetherian, so is B .

(**Hint.** (1) Let's recall that $T_{E/F} : E \rightarrow F, T_{E/F}(e) := \sum_{\sigma \in \text{Embed}_F(E, \overline{E})} \sigma(e)$ is a well-defined map, and $k : E \times E \rightarrow F, k(x, y) := T_{E/F}(xy)$ is a non-degenerate bilinear form. Notice that for every $b \in B$ and $\sigma \in \text{Embed}_F(E, \overline{E})$, $\sigma(b)$ is integral over A . Hence, $T_{E/F}(b)$ is integral over A . Deduce that $T_{E/F}(B) \subseteq A$.

(2) Show that for every $e \in E$, there exists $c \in F$ such that ce is integral over A . Deduce that $E = S^{-1}B$ where $S := F^\times$.

(3) Use the previous step and deduce that there exist a basis $b_1, \dots, b_n$ of E over F such that $b_1, \dots, b_n \in B$.

(4) Use the non-degeneracy of k and show that there exists a dual basis; that means there exists a basis $e_1, \dots, e_n$ of E over F such that $k(e_i, b_i) = 1$ and $k(e_i, b_j) = 0$ if $i \neq j$.

(5) Argue why for all $x \in e$, we have $x = \sum_{i=1}^n k(x, b_i)e_i$.

(6) To show that B is Noetherian, notice that a finitely generated A -module is Noetherian if A is Noetherian. Moreover, a submodule of a Noetherian module is Noetherian.)

2. Suppose G is a totally ordered abelian group, and F is a field. A valuation of F is $v : F \rightarrow G \cup \{\infty\}$ with the following properties:

- (a) $v(a) = \infty \Leftrightarrow a = 0$.
- (b) $\forall g \in G, g < \infty, g + \infty := \infty, \infty + \infty := \infty$.
- (c) $\forall x, y \in F, v(xy) = v(x) + v(y)$.
- (d) $\forall x, y \in F, v(x + y) \geq \min\{v(x), v(y)\}$.

Let $\mathcal{O}_v := \{x \in F \mid v(x) \geq 0\}$ and $\mathfrak{m}_v := \{x \in F \mid v(x) > 0\}$. Prove that $\mathcal{O}_v$ is a valuation ring and $\text{Max}(\mathcal{O}_v) = \{\mathfrak{m}_v\}$.

3. Let A be a valuation ring of a field F . Let $G := F^\times / A^\times$. We say $xA^\times \geq yA^\times$ if $xy^{-1} \in A$. Prove that G is a totally ordered group and

$$v : F \rightarrow G \cup \{\infty\}, \quad v(x) := \begin{cases} xA^\times & \text{if } x \in F^\times \\ \infty & \text{otherwise.} \end{cases}$$

is a valuation of F , and $\mathcal{O}_v = A$.

4. Suppose $f : A \rightarrow B$ is a ring homomorphism, and $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is the corresponding contraction map. Prove that the closure of the image of f^* is $V(\ker f)$. In particular, if f is injective, then the image of f^* is dense.

(**Hint.** Suppose $\overline{f^*(\text{Spec}(B))} = V(\mathfrak{a})$. Deduce that

$$\mathfrak{a} \subseteq \bigcap_{\mathfrak{q} \in \text{Spec}(B)} f^*(\mathfrak{q}) = f^{-1}\left(\bigcap_{\mathfrak{q} \in \text{Spec}(B)} \mathfrak{q}\right) = f^{-1}(\text{Nil}(B)) = \sqrt{\ker f}.$$

When the image of f^* is dense, we say f is dominant.

5. Suppose B is an integral domain, $A \subseteq B$ is a subring, and B is a finitely generated A -algebra. Let $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the contraction map. Prove that the image of f^* contains a dense open subset of $\text{Spec}(A)$.

(**Hint.** Let $b_0 := 1$. Then, there exists $a_0 \in A$ such that for every algebraically closed field Ω , $h_B(\Omega) \xrightarrow{\hat{f}} h_{A_{a_0}}(\Omega)$ is surjective.

For $\mathfrak{p} \in \text{Spec}(A)$, let $\overline{k(\mathfrak{p})}$ be an algebraic closure of the residue field

$$k(\mathfrak{p}) := A_{\mathfrak{p}} / \mathfrak{p}A_{\mathfrak{p}}$$

Recall, that $D_{a_0} := \text{Spec}(A) \setminus V(\langle a_0 \rangle)$ and localization gives us a bijection between D_{a_0} and $\text{Spec}(A_{a_0})$. For $\mathfrak{p} \in D_{a_0}$, $k(\mathfrak{p}) \simeq k(\mathfrak{p}A_{a_0})$, and so we get $\theta_{\mathfrak{p}} \in h_{A_{a_0}}(\overline{k(\mathfrak{p})})$. Hence, there exists a lift $\hat{\theta}_{\mathfrak{p}} \in h_B(\overline{k(\mathfrak{p})})$ of $\theta_{\mathfrak{p}}$. Deduce that $\mathfrak{q} := \ker \hat{\theta}$ is in $\text{Spec}(B)$ and $f^*(\mathfrak{q}) = \mathfrak{p}$. Hence, D_{a_0} is a subset of the image. Notice that $0 \in D_{a_0}$ and $\{0\}$ is dense in $\text{Spec}(A)$.)

6. Suppose k is a field and A is a finitely generated k -algebra. Prove that for every ideal $\mathfrak{a}$ of A ,

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m} \in \text{Max}(A) \cap V(\mathfrak{a})} \mathfrak{m}.$$

(**Hint.** (1) Argue why it is enough to prove the claim when A is the ring of polynomials $k[x_1, \dots, x_n]$ and $\mathfrak{a} = \mathfrak{p}$ is a prime ideal.

(2) Let $\bar{k}$ be an algebraic closure of k . Argue why $\bar{k}[x_1, \dots, x_n]$ is an integral extension of $k[x_1, \dots, x_n]$. Deduce that

$$f^* : \operatorname{Spec}(\bar{k}[x_1, \dots, x_n]) \rightarrow \operatorname{Spec}(k[x_1, \dots, x_n]) \text{ is surjective, and} \\ f^*(\operatorname{Max}(\bar{k}[x_1, \dots, x_n])) = \operatorname{Max}(k[x_1, \dots, x_n]),$$

and so without loss of generality, you can assume that k is algebraically closed.

(3) Use a version of Nullstellensatz to show that if $h \notin \mathfrak{p}$, then there exists a point $p \in k^n$ such that $p \in V_{\mathfrak{p}}(k)$ and $h(p) \neq 0$.

(4) Deduce that for every $h \notin \mathfrak{p}$, there exists $\mathfrak{m} \in \operatorname{Max}(k[x_1, \dots, x_n]) \cap V(\mathfrak{p})$ such that $h \notin \mathfrak{m}$.)

(**Remark.** A ring with this property is called a Hilbert ring or a Jacobson ring.)