

1 Homework 5.

1. Suppose $\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_n$ are prime ideals. Prove that if $\mathfrak{p} \subseteq \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$, then $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some i .

(**Hint.** Prove this by induction on n . Prove the base case $n = 2$. Using the induction hypothesis, assume that for every i , there exists $x_i \in \mathfrak{p} \setminus (\bigcup_{j \neq i} \mathfrak{p}_j)$. Notice that $x_i \in \mathfrak{p}_i$. Consider $x_1 \cdots x_{n-1} + x_n$.)

2. Suppose $f : A \rightarrow B$ is a ring homomorphism. For $\mathfrak{p} \in \text{Spec}(A)$, let

$$S_{\mathfrak{p}} := A \setminus \mathfrak{p}, A_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}A, k(\mathfrak{p}) := A_{\mathfrak{p}}/S_{\mathfrak{p}}^{-1}\mathfrak{p}, \text{ and } B_{\mathfrak{p}} := f(S_{\mathfrak{p}})^{-1}B.$$

- (a) Prove that

$$B \otimes_A k(\mathfrak{p}) \simeq \frac{B_{\mathfrak{p}}}{f(S_{\mathfrak{p}})^{-1}\mathfrak{p}^e}.$$

- (b) Prove that there exists a natural bijection between the fiber $(f^*)^{-1}(\mathfrak{p})$ and $\text{Spec}(\frac{B_{\mathfrak{p}}}{f(S_{\mathfrak{p}})^{-1}\mathfrak{p}^e})$.

(**Hint.** (a) Show that $b \otimes (\frac{a}{s} + S_{\mathfrak{p}}^{-1}\mathfrak{p}) \mapsto \frac{f(a)b}{f(s)} + f(S_{\mathfrak{p}})^{-1}\mathfrak{p}^e$ is a well-defined ring homomorphism. Next, consider the function from $B_{\mathfrak{p}}$ to $B \otimes_A k(\mathfrak{p})$ given by $\frac{b}{f(s)} \mapsto b \otimes \frac{1}{s}$; notice that if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$, then

$$b \otimes \frac{1}{s_1} - b \otimes \frac{1}{s_2} = b \otimes \frac{s_2 - s_1}{s_1 s_2} = bf(s_2 - s_1) \otimes \frac{1}{s_1 s_2} = 0,$$

and so the given function is well-defined. Check that it is a ring homomorphism, and $f(S_{\mathfrak{p}})^{-1}\mathfrak{p}^e$ is in its kernel. Therefore,

$$\frac{b}{f(s)} + f(S_{\mathfrak{p}})^{-1}\mathfrak{p}^e \mapsto b \otimes \frac{1}{s}$$

is a well-defined ring homomorphism. Show that these given ring homomorphisms are inverse of each other.

- (b) Notice that $f^*(\mathfrak{q}) = \mathfrak{p}$ if and only if $\mathfrak{p}^e \subseteq \mathfrak{q}$ and $f(S_{\mathfrak{p}}) \cap \mathfrak{q} = \emptyset$.

(**Remark.** Based on this exercise, you see that there exists a natural bijection between $(f^*)^{-1}(\mathfrak{p})$ and $\text{Spec}(B \otimes_A k(\mathfrak{p}))$. This can be better seen as the fiber product of $\text{Spec}(B)$ and $\text{Spec}(k(\mathfrak{p}))$ over $\text{Spec}(A)$!)

3. Suppose A and B are two unital commutative rings, and $f : A \rightarrow B$ is a ring homomorphism. Consider B as an A -module where $a.b := f(a)b$. Suppose B is a flat A -module. Prove that the following statements are equivalent:

- (a) For every $\mathfrak{a} \trianglelefteq A$, $\mathfrak{a}^{ec} = \mathfrak{a}$.
- (b) $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.
- (c) For every $\mathfrak{m} \in \text{Max}(A)$, $\mathfrak{m}^e \neq B$.
- (d) If M is a non-zero A -module, then $M \otimes_A B \neq 0$.
- (e) For every A -module M , $\theta : M \rightarrow M \otimes_A B, \theta(x) := x \otimes 1$ is injective.

(**Hint.** (a) $\Rightarrow$ (b), in class we proved that $\mathfrak{p}$ is in the image of f^* if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

(b) $\Rightarrow$ (c), if $f^*(\mathfrak{p}') = \mathfrak{m}$, then $\mathfrak{m}^e \subseteq \mathfrak{p}'$.

(c) $\Rightarrow$ (d), For any $x \in M$, $0 \rightarrow Ax \rightarrow M$ is exact. Since B is flat,

$$0 \rightarrow Ax \otimes_A B \rightarrow M \otimes_A B$$

is exact. So to show $M \otimes_A B$ is not zero, it is enough to show $Ax \otimes_A B$ is not zero. Let $\mathfrak{a} := \{a \in A \mid ax = 0\}$; then $Ax \simeq A/\mathfrak{a}$ as an A -module. Hence $Ax \otimes_A B \simeq B/\mathfrak{a}^e$ as an A -module. Suppose $\mathfrak{m}$ is a maximal ideal such that $\mathfrak{a} \subseteq \mathfrak{m}$, and deduce the claim.

(d) $\Rightarrow$ (e), Suppose $M' := \ker \theta$. Since B is a flat A -module,

$$0 \rightarrow M' \otimes_A B \rightarrow M \otimes_A B \xrightarrow{g} (M \otimes_A B) \otimes_A B$$

is exact, where $g := \theta \otimes \text{id}_B$. View $M \otimes_A B$ as an B -module and let

$$h : (M \otimes_A B) \otimes_A B \rightarrow M \otimes_A B, h(x \otimes b) := xb.$$

Show that h is a well-defined B -module homomorphism. Notice that

$$g(m \otimes b) = \theta(m) \otimes b = (m \otimes 1) \otimes b;$$

and so $(h \circ g)(m \otimes b) = (m \otimes 1)b = m \otimes b$. This implies that $h \circ g = \text{id}$. Deduce that g is injective.

(e) $\Rightarrow$ (a) Show that $\bar{f} : A/\mathfrak{a} \rightarrow B/\mathfrak{a}^e, \bar{f}(a + \mathfrak{a}) := f(a) + \mathfrak{a}^e$ is a well-defined injective ring homomorphism.)

4. Suppose $f : A \rightarrow B$ is a ring homomorphism and B is a flat A -module. Suppose $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\mathfrak{p} := \mathfrak{q}^e \in \operatorname{Spec}(A)$. Let $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$ and $S_{\mathfrak{q}} := B \setminus \mathfrak{q}$. Let $A_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}A$, $B_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}B$, and $B_{\mathfrak{q}} := S_{\mathfrak{q}}^{-1}B$. Recall that $B_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ -module. Notice that $B_{\mathfrak{p}}$ can be viewed as a subring of $B_{\mathfrak{q}}$, and it is a localization of $B_{\mathfrak{p}}$, and so $B_{\mathfrak{q}}$ is a flat $B_{\mathfrak{p}}$ -module. Hence, we can view $B_{\mathfrak{q}}$ as a flat $A_{\mathfrak{p}}$ -module. Let $g : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ be the corresponding ring homomorphism. Prove that

$$g^* : \operatorname{Spec}(B_{\mathfrak{q}}) \rightarrow \operatorname{Spec}(A_{\mathfrak{p}})$$

is surjective.

(**Hint.** Use part (c) of the previous problem, and show that $g : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is faithfully flat.)

5. For $\mathfrak{a} \trianglelefteq A$, let $\mathfrak{a}[x] := \{\sum_{i=0}^m a_i x^i \mid a_i \in \mathfrak{a}, m \in \mathbb{Z}^+\}$. Let

$$f : A \hookrightarrow A[x], f(a) := a.$$

- (a) Convince yourself that $\mathfrak{a}^e = \mathfrak{a}[x]$. Show that $\operatorname{Spec} A \xrightarrow{e} \operatorname{Spec}(A[x])$ is a well-defined injection.
 - (b) Prove that, if $\mathfrak{q}$ is a $\mathfrak{p}$ -primary of A , then $\mathfrak{q}^e$ is a $\mathfrak{p}^e$ -primary ideal of $A[x]$.
 - (c) Suppose k is a field. Prove that $0 \subseteq \langle x_1 \rangle \subseteq \cdots \subseteq \langle x_1, \dots, x_n \rangle$ is a chain of prime ideals of $k[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_r \rangle^m$ is $\langle x_1, \dots, x_r \rangle$ -primary for any $1 \leq r \leq n$ and positive integer m .
6. For a multiplicatively closed subset S of a unital commutative ring A and every ideal $\mathfrak{a}$ of A , we let $S(\mathfrak{a}) := \mathfrak{a}^{ec}$ with respect to $A \rightarrow S^{-1}A, a \mapsto \frac{a}{1}$. For a prime ideal $\mathfrak{p}$, let $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$. For a positive integer n , let $\mathfrak{p}^{(n)} := S_{\mathfrak{p}}(\mathfrak{p}^n)$.
- (a) Prove that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$ -primary.
 - (b) Prove that if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$ -primary component.
 - (c) Prove that if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$ -primary component.
 - (d) Prove that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is primary.

- (**Hint.** (a) Notice that $S_{\mathfrak{p}}^{-1}\mathfrak{p}^{(n)} = (S_{\mathfrak{p}}^{-1}\mathfrak{p})^n$, and deduce that $S_{\mathfrak{p}}^{-1}\mathfrak{p}^{(n)}$ is $S_{\mathfrak{p}}^{-1}\mathfrak{p}$ -primary. Hence, $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$ -primary.
- (b) Show that $S(\bigcap_{i=1}^n \mathfrak{q}_i) = \bigcap_{i=1}^n S(\mathfrak{q}_i)$. Notice that if $\mathfrak{q}$ is $\mathfrak{p}$ -primary, then $S_{\mathfrak{p}}(\mathfrak{q}) = \mathfrak{q}$.
- (c) Argue why $S_{\mathfrak{p}}(\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}) = \mathfrak{p}^{(m+n)}$.)