MATH200C, LECTURE 9

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In the previous lecture we proved:

Proposition 1. *Suppose* $\mathfrak{a}$ *is decomposable. Then*

$$\bigcup_{\mathfrak{p}\in \mathrm{Ass}(\mathfrak{a})} \mathfrak{p} = \{x \in A |\ (\mathfrak{a} : x) \neq \mathfrak{a}\}.$$

We have also pointed out the connection with zero-divisors; let's add a bit to this connection:

Corollary 2. $D(A) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(0)} \mathfrak{p}$ *and* $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(0)} \mathfrak{p}$.

Proof. Notice that $x \in D(A)$ if and only if $(0 : x) \neq 0$; and so $D(A) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(0)} \mathfrak{p}$, by the previous proposition.

Since 0 is decomposable, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, there is $\mathfrak{p}' \in \mathrm{Ass}(0)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Hence $\bigcap_{\mathfrak{p}'\in \mathrm{Ass}(0)} \mathfrak{p}' \subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$; this implies that

$$\bigcap_{\mathfrak{p}'\in \mathrm{Ass}(0)} \mathfrak{p}' \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Spec}(A)} \mathfrak{p} = \mathrm{Nil}(A).$$

Clearly $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p}\in \mathrm{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p}'\in \mathrm{Ass}(0)} \mathfrak{p}'$; and claim follows. □

Localization and primary ideals

Suppose S is a multiplicatively closed subset of A; and let $f : A \to S^{-1}A$, $f(a) := a/1$. We have proved that any ideal of $S^{-1}A$ is an extended ideal; and so $\widetilde{\mathfrak{a}}^{ce} = \widetilde{\mathfrak{a}}$. Not any ideal of A is a contracted ideal. Now let $S(\mathfrak{a}) := \mathfrak{a}^{ec}$. Recall that

Lemma 3. *(1)* $S(\mathfrak{a}) \supseteq \mathfrak{a}$ *for any* $\mathfrak{a} \trianglelefteq A$.
(2) $S^{-1}S(\mathfrak{a}) = S^{-1}\mathfrak{a}$.
(3) $S(\mathfrak{p}) = \mathfrak{p}$ *if* $\mathfrak{p} \in \mathrm{Spec}(A)$ *and* $\mathfrak{p} \cap S = \varnothing$.

Proof. (1) $S(\mathfrak{a}) = \mathfrak{a}^{ec} \supseteq \mathfrak{a}$. (2) $S^{-1}(S(\mathfrak{a})) = \mathfrak{a}^{ece} = \mathfrak{a}^{e} = S^{-1}\mathfrak{a}$. (3) We have proved earlier. □

Let us have a better understanding of $S(\mathfrak{a})$:

Lemma 4. *$S(\mathfrak{a}) = \bigcup_{s\in S}(\mathfrak{a} : s)$.*

Proof.

$$\begin{aligned} a \in S(\mathfrak{a}) \Leftrightarrow & \frac{a}{1} \in S^{-1}\mathfrak{a} \Leftrightarrow \exists s' \in S, a' \in \mathfrak{a}, \frac{a}{1} = \frac{a'}{s'}, \\ \Leftrightarrow & \exists s'' \in S, s''(s'a - a') = 0 \Leftrightarrow \underbrace{(s''s')}_{s} a = s''a' \in \mathfrak{a} \\ \Rightarrow & \exists s \in S, sa \in \mathfrak{a} \Rightarrow a \in \bigcup_{s\in S}(\mathfrak{a} : s). \end{aligned}$$

$$\begin{aligned} a \in \bigcup_{s\in S}(\mathfrak{a} : s) \Rightarrow & \exists s \in S, sa \in \mathfrak{a} \Rightarrow \frac{a}{1} = \frac{sa}{s} \in S^{-1}\mathfrak{a} \\ \Rightarrow & a \in S(\mathfrak{a}); \end{aligned}$$

and claim follows. □

Proposition 5. *(1) If $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S \neq \varnothing$, then $S^{-1}\mathfrak{q} = S^{-1}A$.*
(2) If $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \varnothing$, then $S(\mathfrak{q}) = \mathfrak{q}$.
(3) If $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \varnothing$, then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.
(4) Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{p} \cap S$. Then extension and contraction maps induce bijections between

$$\{\mathfrak{q} \trianglelefteq A |\ \mathfrak{q} \textit{ is } \mathfrak{p}\textit{-primary}\} \textit{ and } \{\widetilde{\mathfrak{q}} \trianglelefteq S^{-1}A |\ \widetilde{\mathfrak{q}} \textit{ is } S^{-1}\mathfrak{p}\textit{-primary}\}.$$

Proof. (1) If $a \in \mathfrak{p} \cap S$, then for some positive integer n, $a^n \in \mathfrak{q} \cap S$; and so $S^{-1}\mathfrak{q} = S^{-1}A$.

(2) Suppose $a \in S(\mathfrak{q})$; then there is $s \in S$ such that $sa \in \mathfrak{q}$. Since $S \cap \mathfrak{p} = \varnothing$, $s \notin \mathfrak{p}$. As $\mathfrak{q}$ is $\mathfrak{p}$-primary, $sa \in \mathfrak{q}$, and $s \notin \mathfrak{p}$, we deduce that $a \in \mathfrak{q}$.

(3) First we show that $\sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p}$.

$$\begin{aligned} \frac{a}{s} \in \sqrt{S^{-1}\mathfrak{q}} \Rightarrow & \exists n \in \mathbb{Z}^+, \left(\frac{a}{s}\right)^n \in S^{-1}\mathfrak{q}, \Rightarrow \left(\frac{a}{1}\right)^n \in S^{-1}\mathfrak{q}, \\ \Rightarrow & a^n \in S(\mathfrak{q}) = \mathfrak{q} \Rightarrow a \in \sqrt{\mathfrak{q}} = \mathfrak{p} \\ \Rightarrow & \frac{a}{s} \in S^{-1}\mathfrak{p}. \end{aligned}$$

And

$$\frac{a}{s} \in S^{-1}\mathfrak{p} \Rightarrow \frac{a}{1} \in S^{-1}\mathfrak{p} \Rightarrow a \in S(\mathfrak{p}) = \mathfrak{p}$$
$$\Rightarrow \exists n \in \mathbb{Z}^+, a^n \in \mathfrak{q} \Rightarrow \frac{a}{s} \in \sqrt{S^{-1}\mathfrak{q}}.$$

Next we show $S^{-1}\mathfrak{q}$ is primary; we have already proved that it is proper. Suppose $\frac{a}{s} \cdot \frac{a'}{s'} \in S^{-1}\mathfrak{q}$ and $\frac{a'}{s'} \notin \sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p}$. Then for some $s'' \in S$, we have $s''aa' \in \mathfrak{q}$ and $a' \notin \mathfrak{p}$. Since $S \cap \mathfrak{p} = \varnothing$, $s'' \in S$, $a' \notin \mathfrak{p}$, and $\mathfrak{p}$ is prime, we deduce that $s''a' \notin \mathfrak{p}$. Since $\mathfrak{q}$ is $\mathfrak{p}$-primary, $s''a' \notin \mathfrak{p}$, and $(s''a')a \in \mathfrak{q}$, we have $a \in \mathfrak{q}$. Thus $\frac{a}{s} \in S^{-1}\mathfrak{q}$.

(4) To show this part, it remains to show $\mathfrak{q} := \widetilde{\mathfrak{q}}^c$ is $\mathfrak{p}$-primary if $\widetilde{\mathfrak{q}}$ is $S^{-1}\mathfrak{p}$-primary. First we show that $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

Suppose $a \in \sqrt{\mathfrak{q}}$; then for some positive integer n, $a^n \in \mathfrak{q}$, which means $\frac{a^n}{1} \in \widetilde{\mathfrak{q}}$. Hence $\frac{a}{1} \in \sqrt{\widetilde{\mathfrak{q}}} = S^{-1}\mathfrak{p}$. Therefore $a \in S(\mathfrak{p}) = \mathfrak{p}$. If $b \in \mathfrak{p}$, then $\frac{b}{1} \in S^{-1}\mathfrak{p}$. Hence for some positive integer n, $\frac{b^n}{1} \in \widetilde{\mathfrak{q}}$, which implies that $b^n \in \mathfrak{q}$. Thus $b \in \sqrt{\mathfrak{q}}$.

Next we show that $\mathfrak{q}$ is primary. We have already showed that it is proper. Suppose $xy \in \mathfrak{q}$ and $y \notin \sqrt{\mathfrak{q}} = \mathfrak{p}$. Then $\frac{x}{1} \cdot \frac{y}{1} \in \widetilde{\mathfrak{q}}$ and $\frac{y}{1} \notin S^{-1}\mathfrak{p}$ (as $S(\mathfrak{p}) = \mathfrak{p}$). As $\widetilde{\mathfrak{q}}$ is primary, we deduce that $\frac{x}{1} \in \widetilde{\mathfrak{q}}$. Hence $x \in \mathfrak{q}$. □

Localization and primary decomposition

Lemma 6. *Suppose $\bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition of $\mathfrak{a}$, $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, and $S \cap \mathfrak{p}_i = \varnothing$ if and only if $1 \le i \le m$. Then*

(1) $S^{-1}\mathfrak{a} = \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i$;
(2) $S(\mathfrak{a}) = \bigcap_{i=1}^m \mathfrak{q}_i$.

Proof. For any i, $\mathfrak{a}_i \subseteq \mathfrak{q}_i$; and so $S^{-1}\mathfrak{a} \subseteq S^{-1}\mathfrak{q}_i$ and $S(\mathfrak{a}) \subseteq S(\mathfrak{q}_i)$. Therefore

$$S^{-1}\mathfrak{a} \subseteq \bigcap_{i=1}^n S^{-1}\mathfrak{q}_i \text{ and } S(\mathfrak{a}) \subseteq \bigcap_{i=1}^n S(\mathfrak{q}_i).$$

By the previous lemma, $S^{-1}\mathfrak{q}_i = S^{-1}A$ and $S(\mathfrak{q}_i) = A$ if $i > m$, and $S(\mathfrak{q}_i) = \mathfrak{q}_i$ if $1 \le i \le m$; hence

$$S^{-1}\mathfrak{a} \subseteq \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i \text{ and } S(\mathfrak{a}) \subseteq \bigcap_{i=1}^m \mathfrak{q}_i.$$

Suppose $\frac{a}{s} \in \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i$; then $\frac{a}{1} \in \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i$. Hence $a \in \bigcap_{i=1}^m S(\mathfrak{q}_i) = \bigcap_{i=1}^m \mathfrak{q}_i$. For $i > m$, let $s_i \in S \cap \mathfrak{q}_i$; set $s' := \prod_{i>m} s_i$. Then $s'a \in \bigcap_{i=1}^n \mathfrak{q}_i = \mathfrak{a}$ and

$s' \in S$. Therefore $\frac{a}{s} = \frac{s'a}{s's} \in S^{-1}\mathfrak{a}$. This implies that $S^{-1}\mathfrak{a} = \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i$. After contraction we get

$$S(\mathfrak{a}) = \bigcap_{i=1}^{m} S(\mathfrak{q}_i) = \bigcap_{i=1}^{m} \mathfrak{q}_i.$$

$\square$

The second uniqueness theorem

Definition 7. *Suppose* $\mathfrak{a}$ *is decomposable. A subset* Σ *of* $\mathrm{Ass}(\mathfrak{a})$ *is called* isolated *if* Σ *is not empty and*

$$(\mathfrak{p} \in \Sigma, \mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a}), \mathfrak{p}' \subseteq \mathfrak{p}) \Rightarrow \mathfrak{p}' \in \Sigma.$$

An important example is the following: suppose $\mathfrak{p}$ is a minimal element of $\mathrm{Ass}(\mathfrak{a})$; then $\{\mathfrak{p}\}$ is isolated.

Theorem 8. *Suppose* $\mathfrak{a}$ *is decomposable and* $\Sigma \subseteq \mathrm{Ass}(\mathfrak{a})$ *is isolated. Suppose* $\bigcap_{i=1}^n \mathfrak{q}_i$ *is a reduced primary decomposition. Then* $\bigcap_{\sqrt{\mathfrak{q}_i} \in \Sigma} \mathfrak{q}_i$ *just depends on* $\mathfrak{a}$ *and* Σ *and it is independent of the choice of a primary decomposition. In particular, if* $\mathfrak{p}$ *is a minimal element of* $\mathrm{Ass}(\mathfrak{a})$*, then there is a* $\mathfrak{p}$*-primary ideal* $\mathfrak{q}$ *that appears in all the reduced primary decompositions of* $\mathfrak{a}$*.*

Proof. We want to use a suitable localization and use the previous lemma. So we need to find a multiplicatively closed set S_Σ such that $S_\Sigma(\mathfrak{a})$ becomes $\bigcap_{\mathfrak{p}_i \in \Sigma} \mathfrak{q}_i$ where $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. By the previous lemma, $S_\Sigma(\mathfrak{a}) = \bigcap_{\mathfrak{p}_i \cap S_\sigma = \varnothing} \mathfrak{q}_i$. So we have to find S_Σ in a way that

$$S_\Sigma \cap \mathfrak{p}_i = \varnothing \Leftrightarrow \mathfrak{p}_i \in \Sigma.$$

This suggests that we let $S_\Sigma := A \setminus \bigcap_{\mathfrak{p} \in \Sigma} \mathfrak{p}$.

Claim 1. $S_\sigma := A \setminus \bigcap_{\mathfrak{p} \in \Sigma} \mathfrak{p}$ is multiplicatively closed.

Proof of Claim 1. Suppose $s_1, s_2 \in S_\Sigma$ and $s_1 s_2 \notin S_\Sigma$; then $s_1 s_2 \in \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$. Since $\mathfrak{p}$ is prime, either $s_1 \in \mathfrak{p}$ or $s_2 \in \mathfrak{p}$. Hence either $s_1 \notin S_\Sigma$ or $s_2 \notin S_\Sigma$; this is a contradiction.

Claim 2. $S_\Sigma \cap \mathfrak{p}_i = \varnothing \Leftrightarrow \mathfrak{p}_i \in \Sigma$.

Proof of Claim 2. If $\mathfrak{p}_i \in \Sigma$, then we have that $S_\Sigma \cap \mathfrak{p}_i = \varnothing$, by definition of S_Σ. Next suppose $\mathfrak{p}_i \cap S_\Sigma = \varnothing$; then

$$\mathfrak{p}_i \subseteq \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}.$$

By a result that we have proved earlier (and you have proved its generalization, McCoy's result), we have that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$. Since Σ is isolated, we deduce that $\mathfrak{p}_i \in \Sigma$.

Overall we get $S_\Sigma(\mathfrak{a}) = \bigcap_{\mathfrak{p}_i \in \Sigma} \mathfrak{q}_i$. Hence this intersection just depends on $\mathfrak{a}$ and Σ and it is independent of the choice of a reduced primary factorization. $\square$

Krull dimension one integral domains

Definition 9. *The Krull dimension of a ring A is defined to be*

$$\dim A := \sup\{n \in \mathbb{Z}^{\geq 0} | \exists \mathfrak{p}_i \in \operatorname{Spec}(A), \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.$$

Example 10. *Suppose D is an integral domain; then $\dim D = 0$ if and only if D is a field.*

Proof. ($\Rightarrow$) Suppose to the contrary that D is not a field. So there is $d \in D \setminus D^\times \cup \{0\}$; this means $\langle d \rangle$ is a non-zero proper ideal of D. So there is a non-zero maximal ideal $\mathfrak{m}$ of D (that contains d). This means $\dim D \geq 1$ as $0 \subsetneq \mathfrak{m}$ is a chain of prime ideals of D; and this is a contradiction.

($\Leftarrow$) Since D is a field, its only proper ideal is 0; and claim follows. $\square$

Example 11. *Suppose D is an integral domain which is not a field. Then $\dim D = 1$ if and only if $\operatorname{Spec}(D) = \{0\} \cup \operatorname{Max}(D)$.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{p}$ is a non-zero prime ideal, and $\mathfrak{m}$ is a maximal ideal of D that contains $\mathfrak{p}$ as a subset. Then $0 \subsetneq \mathfrak{p} \subseteq \mathfrak{m}$ is a chain of prime ideals. As $\dim D = 1$, we deduce that $\mathfrak{p} = \mathfrak{m}$, which means $\mathfrak{p}$ is a maximal ideal.

($\Leftarrow$) Since $\operatorname{Spec}(D) = \{0\} \cup \operatorname{Max}(D)$, any non-zero prime ideal is maximal. This means we cannot have a chain of prime ideals that have length two (that means there are no prime ideals such that $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ as $\mathfrak{p}_1$ is maximal). Thus $\dim D \leq 1$. By the previous example and the assumption that D is not a field, we deduce that $\dim D \neq 0$. Hence $\dim D = 1$. $\square$

Proposition 12. *Suppose D is an integral domain of Krull dimension 1; then any decomposable ideal has a unique primary decomposition.*

Proof. Notice that 0 is a reduced primary decomposition of 0; hence $\operatorname{Ass}(0) = \{0\}$. Since 0 is minimal in $\operatorname{Ass}(0)$, 0 is the only reduced primary decomposition of 0.

Suppose $\mathfrak{a}$ is a non-zero decomposable ideal. Then $0 \notin \mathrm{Ass}(\mathfrak{a})$. By the previous lemma, we deduce that $\mathrm{Ass}(\mathfrak{a}) \subseteq \mathrm{Max}(A)$. This implies that any element of $\mathrm{Ass}(\mathfrak{a})$ is minimal in $\mathrm{Ass}(\mathfrak{a})$ (one maximal ideal cannot be a subset of another maximal ideal). Hence for any $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})$, there is a $\mathfrak{p}$-primary ideal $\mathfrak{q}_{\mathfrak{p}}$ that appears in all the reduced primary decompositions of $\mathfrak{a}$. Hence $\bigcap_{\mathfrak{p}\in \mathrm{Ass}(\mathfrak{a})} \mathfrak{q}_{\mathfrak{p}}$ is the unique reduced primary decomposition of $\mathfrak{a}$. $\square$