MATH200C, LECTURE 6

GOLSEFIDY

Zariski topology

In the previous lecture we were proving the following.

Proposition 1 (Basics of divisibility for ideals). *(1)* $\mathfrak{a}|\mathfrak{b} \Rightarrow V(\mathfrak{a}) \subseteq V(\mathfrak{b})$.

(2) $\gcd(\{\mathfrak{a}_i\}_{i\in I}) = \sum_{i\in I} \mathfrak{a}_i$ *and* $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} V(\mathfrak{a}_i)$.

(3) $\operatorname{lcm}(\{\mathfrak{a}_i\}_{i\in I}) = \bigcap_{i\in I} \mathfrak{a}_i$ *and* $V(\bigcap_{i=1}^n \mathfrak{a}_i) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$*; for an infinite family of ideals equality does not necessarily hold.*

(4) $V(A) = \varnothing$ *and* $V(0) = \operatorname{Spec}(A)$.

The rest of proof. (3) $\forall i \in I, \mathfrak{a}_i|\mathfrak{b} \Leftrightarrow \forall i \in I, \mathfrak{b} \subseteq \mathfrak{a}_i \Leftrightarrow \mathfrak{b} \subseteq \bigcap_i \mathfrak{a}_i \Leftrightarrow (\bigcap_i \mathfrak{a}_i)|\mathfrak{b}$. Since $\mathfrak{a}_i | \bigcap_j \mathfrak{a}_j$, $V(\mathfrak{a}_i) \subseteq V(\bigcap_j \mathfrak{a}_j)$; and so $\bigcup_{i\in I} V(\mathfrak{a}_i) \subseteq V(\bigcap_{i\in I} \mathfrak{a}_i)$. Suppose to the contrary that $\mathfrak{p} \in V(\bigcap_{i=1}^n \mathfrak{a}_i) \setminus \bigcap_{i=1}^n V(\mathfrak{a}_i)$; then $\bigcap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$ and for any i there is $a_i \in \mathfrak{a}_i \setminus \mathfrak{p}$. Therefore $\prod_{i=1}^n a_i \notin \mathfrak{p}$ and $\prod_{i=1}^n a_i \in \bigcap_{i=1}^n \mathfrak{a}_i$, which contradicts $\bigcap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$.

Let $\mathscr{P}$ be the set of prime numbers in $\mathbb{Z}$. Then $\bigcap_{p\in\mathscr{P}} p\mathbb{Z} = 0$; and so 0 is in $V(\bigcap_{p\in\mathscr{P}} p\mathbb{Z})$; but 0 is not in $\bigcup_{p\in\mathscr{P}} V(p\mathbb{Z})$.

(4) is clear. $\square$

Definition 2 (Zariski topology). *Let* $\{V(\mathfrak{a})\}_{\mathfrak{a}\trianglelefteq A}$ *be the set of closed subsets of* $\operatorname{Spec}(A)$*. The above proposition shows that this collection of closed sets give us a well-defined topology on* $\operatorname{Spec}(A)$*. This is called the Zariski topology of* $\operatorname{Spec}(A)$*.*

Before we continue studying the connection between Zariski topology and algebraic properties of the ambient ring, let us prove a technical lemma that will be needed later. This lemma shows us how union of ideals is far from being an ideal. You will see a strengthening of this result in your HW assignment.

Proposition 3. *Suppose* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(A)$*,* $\mathfrak{a} \trianglelefteq A$*, and* $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$*. Then* $\mathfrak{a} \subseteq \mathfrak{p}_i$ *for some* i*.*

Proof. We proceed by induction on n. So W.L.O.G. we can assume that $\mathfrak{a} \not\subseteq \bigcup_{i\neq i_0} \mathfrak{p}_i$ for any $i_0 \in [1..n]$. Let $a_{i_0} \in \mathfrak{a} \backslash \bigcup_{i\neq i_0} \mathfrak{p}_i$ for any $i_0 \in [1..n]$. Since $\mathfrak{a} \subseteq \bigcup_i \mathfrak{p}_i$, we have that $a_i \in \mathfrak{p}_i$. Let $a := \prod_{i=1}^{n-1} a_i + a_n$. Notice that $a' := \prod_{i=1}^{n-1} a_i \in \bigcap_{i=1}^{n-1} \mathfrak{p}_i$ as $a_i \in \mathfrak{p}_i$ and $a' = \prod_{i=1}^{n-1} a_i \notin \mathfrak{p}_n$ as $a_i \notin \mathfrak{p}_n$ and $\mathfrak{p}_n$ is a prime ideal. On the other hand, $a := a' + a_n \in \mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$. So either $a \in \mathfrak{p}_n$ or $a \in \mathfrak{p}_i$ for some $i \leq n-1$.

$$\begin{cases} a \in \mathfrak{p}_n \Rightarrow a' + a_n \in \mathfrak{p}_n \xRightarrow{a_n \in \mathfrak{p}_n} a' \in \mathfrak{p}_n, \text{which is a contradiction.} \\ a \in \mathfrak{p}_i \Rightarrow a' + a_n \in \mathfrak{p}_i \xRightarrow{a' \in \mathfrak{p}_i} a_n \in \mathfrak{p}_i, \text{which is a contradiction.} \end{cases}$$

□

Now we go back to understanding Zariski-topology.

Lemma 4. *(1) Any non-empty closed set of* $\mathrm{Spec}(A)$ *intersects* $\mathrm{Max}(A)$*.*

(2) $\{\mathfrak{m} \in \mathrm{Spec}(A) | \ \mathfrak{m}$ *is a closed point in* $\mathrm{Spec}(A)\} = \mathrm{Max}(A)$*.*

(3) In an integral domain D, 0 is dense in $\mathrm{Spec}(A)$ *(that is why it is called the generic point of* $\mathrm{Spec}(A)$*).*

Proof. (1) If $V(\mathfrak{a}) \neq \varnothing$, then $\mathfrak{a}$ is a proper ideal. Hence there is a maximal ideal $\mathfrak{m}$ that contains $\mathfrak{a}$ as a subset. So $\mathfrak{m} \in V(\mathfrak{a})$.

(2) If $\mathfrak{m}$ is a closed point in $\mathrm{Spec}(A)$, then there is an ideal $\mathfrak{a}$ such that $V(\mathfrak{a}) = \{\mathfrak{m}\}$. Now by part (1), $\mathfrak{m} \in \mathrm{Max}(A)$.

If $\mathfrak{m} \in \mathrm{Max}(A)$, then clearly $V(\mathfrak{m}) = \{\mathfrak{m}\}$.

(3) is clear. □

Contraction and extension of ideals

Lemma 5. *Suppose $f : A \to B$ is a ring homomorphism. For $\mathfrak{b} \trianglelefteq B$, let $\mathfrak{b}^c := f^{-1}(\mathfrak{b})$, and for $\mathfrak{a} \trianglelefteq A$, let $\mathfrak{a}^e := \langle f(\mathfrak{a}) \rangle$. Then $\mathfrak{b}^c$ is an ideal of A and $\mathfrak{a}^e$ is an ideal of A.*

Proof. It is clear. □

In the above setting $\mathfrak{b}^c$ is called the contraction of $\mathfrak{b}$ and $\mathfrak{a}^e$ is called the extension of $\mathfrak{a}$. For any ring A, let ideal(A) be the set of its ideals. So $\mathfrak{b} \mapsto \mathfrak{b}^c$ gives us a function ideal(B) $\to$ ideal(A) and $\mathfrak{a} \mapsto \mathfrak{a}^e$ gives us a function ideal(A) $\to$ ideal(B).

Lemma 6. *In the above setting, we have:*

(1) $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$ *and* $\mathfrak{a}^{ec} \supseteq \mathfrak{a}$*.*

(2) $\mathfrak{b}^{cec} = \mathfrak{b}^c$ *and* $\mathfrak{a}^{ece} = \mathfrak{a}^e$*.*

(3) The contraction and extension maps induce bijections between the set of contracted ideals and extended ideals.

Proof. (1) Since $f(f^{-1}(\mathfrak{b})) \subseteq \mathfrak{b}$, $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$. Since $f(\mathfrak{a}) \subseteq \mathfrak{a}^e$,

$$\mathfrak{a} \subseteq f^{-1}(f(\mathfrak{a})) \subseteq f^{-1}(\mathfrak{a}^e) = \mathfrak{a}^{ec}.$$

(2) $\mathfrak{b}^{cec} = (\mathfrak{b}^c)^{ec} \supseteq \mathfrak{b}^c$ and $\mathfrak{b}^{cec} = (\mathfrak{b}^{ce})^c \subseteq \mathfrak{b}^c$; and so $\mathfrak{b}^{cec} = \mathfrak{b}^c$. $\mathfrak{a}^{ece} = (\mathfrak{a}^{ec})^e \supseteq \mathfrak{a}^e$ and $\mathfrak{a}^{ece} = (\mathfrak{a}^e)^{ce} \subseteq \mathfrak{a}^e$; and so $\mathfrak{a}^{ece} = \mathfrak{a}^e$. (3) is clear because of (2). □

Lemma 7. *In the above setting, $A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$ for any $\mathfrak{b} \in$ ideal(B).*

Proof. Let $\overline{f} : A \to B/\mathfrak{b}, \overline{f}(a) := f(a) + \mathfrak{b}$. Then $\ker \overline{f} = \mathfrak{b}^c$; and so by the first isomorphism theorem claim follows. □

Proposition 8. *Suppose $f : A \to B$ is a ring homomorphism. Then*

$$f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A), f^*(\mathfrak{q}) := \mathfrak{q}^c$$

is a continuous map.

Proof. By the previous lemma, $A/f^*(\mathfrak{q})$ can be embedded into $B/\mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, $B/\mathfrak{q}$ is an integral domain. Hence $A/f^*(\mathfrak{q})$ is an integral domain (notice that $f(1_A) = 1_B$ and so $f^*(\mathfrak{q})$ is a proper ideal). Therefore $f^*(\mathfrak{q})$ is a prime ideal. We also have

$$f^*(\mathfrak{q}) \in V(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \mathfrak{q}^c \Leftrightarrow \mathfrak{a}^e \subseteq \mathfrak{q} \Leftrightarrow \mathfrak{q} \in V(\mathfrak{a}^e),$$

which means $(f^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$; and so preimage of a closed set under f^* is closed. Therefore f^* is continuous. □

Lemma 9. *Let $\pi : A \to A/\mathfrak{a}, \pi(a) := a + \mathfrak{a}$. Then π^* induces a homeomorphism between* $\operatorname{Spec}(A/\mathfrak{a})$ *and* $V(\mathfrak{a})$.

Proof. Suppose $\overline{\mathfrak{p}} \in \operatorname{Spec}(A/\mathfrak{a})$; then $\mathfrak{a} = \ker \pi \subseteq \pi^*(\overline{\mathfrak{p}})$. Hence $\pi^*(\overline{\mathfrak{p}}) \in V(\mathfrak{a})$.

If $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p}/\mathfrak{a} \in$ ideal(A/$\mathfrak{a}$) and $(A/\mathfrak{a})/(\mathfrak{p}/\mathfrak{a}) \simeq A/\mathfrak{p}$ is an integral domain. Hence $\overline{\mathfrak{p}} := \mathfrak{p}/\mathfrak{a} \in \operatorname{Spec}(A/\mathfrak{a})$; and $\mathfrak{p} = \pi^*(\overline{\mathfrak{p}})$.

If $\mathfrak{p} := \pi^*(\overline{\mathfrak{p}}_1) = \pi^*(\overline{\mathfrak{p}}_2)$, then $\overline{\mathfrak{p}}_i = \mathfrak{p}/\mathfrak{a}$ for any i; and so π^* is injective.

For any $\overline{\mathfrak{b}} \in$ ideal(A/$\mathfrak{a}$), $\overline{\mathfrak{b}} = \overline{\mathfrak{b}}^c/\mathfrak{a}$. Hence

$$\overline{\mathfrak{p}} \in V(\overline{\mathfrak{b}}) \Leftrightarrow \overline{\mathfrak{b}} \subseteq \overline{\mathfrak{p}} \Leftrightarrow \overline{\mathfrak{b}}^c/\mathfrak{a} \subseteq \overline{\mathfrak{p}}^c/\mathfrak{a} \Leftrightarrow \overline{\mathfrak{b}}^c \subseteq \overline{\mathfrak{p}}^c \Leftrightarrow f^*(\overline{\mathfrak{p}}) \in V(\overline{\mathfrak{b}}^c),$$

which implies f^* is a closed map; and claim follows. □

Lemma 10. *Let* $f : A \to S^{-1}A, f(a) := a/1$. *Then for any* $\mathfrak{a} \in \mathrm{ideal(A)}, \mathfrak{a}^{\mathrm{e}} = \mathrm{S}^{-1}\mathfrak{a}$ *and for any* $\widetilde{\mathfrak{a}} \in \mathrm{ideal(S^{-1}A)}$, $\widetilde{\mathfrak{a}} = \widetilde{\mathfrak{a}}^{ce}$.

Proof. For any $s \in S$ and $a \in \mathfrak{a}$, $a/s = (1/s)(a/1) \in \mathfrak{a}^e$; and so $S^{-1}\mathfrak{a} \subseteq \mathfrak{a}^e$. We have seen that $S^{-1}\mathfrak{a}$ is an ideal of $S^{-1}A$. Hence $\mathfrak{a}^e = S^{-1}A$.

Let $\mathfrak{a} := \widetilde{\mathfrak{a}}^c$. Then $\widetilde{\mathfrak{a}} \supseteq \mathfrak{a}^e$. On the other hand,

$$a/s \in \widetilde{\mathfrak{a}} \Rightarrow (s/1)(a/s) \in \widetilde{\mathfrak{a}} \Rightarrow a/1 \in \widetilde{\mathfrak{a}} \Rightarrow a \in \widetilde{\mathfrak{a}}^c = \mathfrak{a} \Rightarrow a/s \in S^{-1}\mathfrak{a} = \mathfrak{a}^e;$$

and claim follows. □

Corollary 11. *Let* $f : A \to S^{-1}A, f(a) := a/1$. *Then contraction is an injective map* $\mathrm{ideal(S^{-1}A)} \to \mathrm{ideal(A)}$ *and extension is a surjective map* $\mathrm{ideal(A)} \to \mathrm{ideal(S^{-1}A)}$

Lemma 12. *Let* $f : A \to S^{-1}A, f(a) := a/1$. *Then* f^* *induces a bijection between* $\mathrm{Spec}(S^{-1}A)$ *and* $\{\mathfrak{p} \in \mathrm{Spec}(A) | \, \mathfrak{p} \cap S = \varnothing\}$. *Moreover if* $\mathfrak{p} \in \mathrm{Spec}(A)$ *and* $\mathfrak{p} \cap S = \varnothing$, *then* $\mathfrak{p}^{ec} = \mathfrak{p}$.

Proof. Step 1. By the above Corollary, contraction is injective; and so f^* is injective.

Step 2. If $f^*(\widetilde{\mathfrak{p}}) = \mathfrak{p}$, then by the above Lemma $\widetilde{\mathfrak{p}} = S^{-1}\mathfrak{p}$; in particular, $S^{-1}\mathfrak{p}$ is a proper ideal. This implies $S \cap \mathfrak{p} = \varnothing$.

Step 3. Suppose $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{p} \cap S = \varnothing$. Then $A/\mathfrak{p}$ is an integral domain and $\overline{S} := \pi(S)$ does not contain 0, where $\pi : A \to A/\mathfrak{p}, \pi(a) := a + \mathfrak{p}$. Hence $\overline{S}^{-1}(A/\mathfrak{p})$ can be embedded into the field $Q(A/\mathfrak{p})$ of fractions of $A/\mathfrak{p}$. Let θ be the composite $A \to A/\mathfrak{p} \hookrightarrow Q(A/\mathfrak{p})$ homomorphism.

We will continue next time. □

A question asked during lecture: Is it true that $\overline{\bigcup_{i\in I} V(\mathfrak{a}_i)} = V(\bigcap_{i\in I} \mathfrak{a}_i)$?

Let $A := \mathbb{Z}$ and $\mathfrak{a}_i := 2^i\mathbb{Z}$. Then $V(\mathfrak{a}_i) = \{2\mathbb{Z}\}$ for any i and $\bigcap_i \mathfrak{a}_i = 0$; and so $V(\bigcap_i \mathfrak{a}_i) = \mathrm{Spec}(\mathbb{Z})$. Hence $\bigcup_i V(\mathfrak{a}_i) = \{2\mathbb{Z}\}$ is closed and is not equal to $V(\bigcap_i \mathfrak{a}_i)$.