MATH200C, LECTURE 0

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Review of main results in field theory.

Let's recall some of the main results and concepts in field theory that we have covered so far:

(1) E/F and K/E are finite extensions $\Leftrightarrow$ K/F is a finite extension.
(2) E/F and K/E are algebraic extensions $\Leftrightarrow$ K/F is an algebraic extension.
(3) $\{\alpha \in E |\ \alpha \text{ is algebraic over } F\}$ is an algebraic extension of F if E/F is an extension; it is called the algebraic closure of F in E.
(4) Suppose $\sigma : F_1 \to F_2$ is a field isomorphism, $p(x) \in F_1[x]$ is irreducible, E_1 is a splitting field of $p(x)$ over F_1, and E_2 is a splitting field of $\sigma(p)$ over F_2. Suppose $\alpha_1 \in E_1$ is a zero of p and $\alpha_2 \in E_2$ is a zero of $\sigma(p)$. Then there is a field isomorphism $\widehat{\sigma} : F_1[\alpha_1] \to F_2[\alpha_2]$ such that $\widehat{\sigma}|_{F_1} = \sigma$.
(5) Suppose $\sigma : F_1 \to F_2$ is a field isomorphism, $f(x) \in F_1[x]$, E_1 is a splitting field of $f(x)$ over F_1 and E_2 is a splitting field of $\sigma(f)$ over F_2. Then there is a field isomorphism $\widehat{\sigma} : E_1 \to E_2$ such that $\widehat{\sigma}|_{F_1} = \sigma$.
(6) For any field F, there is an algebraically closed field $\overline{F}$ such that $\overline{F}/F$ is an algebraic extension; this is called an algebraic closure of F.
(7) If $\sigma : F_1 \to F_2$ is a field isomorphism and $\overline{F}_i$ is an algebraic closure of F_i, then there is a field isomorphism $\widehat{\sigma} : \overline{F}_1 \to \overline{F}_2$ such that $\widehat{\sigma}|_{F_1} = \sigma$.
(8) Suppose $F \subseteq E \subseteq \overline{F}$ is a tower of fields and $\overline{F}$ is an algebraic closure of F. Then E/F is called a normal extension if the following equivalent properties hold:
   (a) For any $\sigma \in \mathrm{Aut}(\overline{F}/F)$, $\sigma(E) = E$.
   (b) For any $\alpha \in E$, $m_{\alpha,F}(x)$ can be written as a product of degree 1 factors in $E[x]$.
   (c) E is a splitting field of a family $\mathscr{F} \subseteq F[x]$ of polynomials over F.
   (d) There are $\{E_i\}_{i\in I}$ and $\{p_i\}_{i\in I} \subseteq F[x]$ such that
      (i) E_i is a splitting field of $p_i(x)$ over F.

(ii) For any $i, j \in I$, there is $k \in I$ such that $E_i \cup E_j \subseteq E_k$.

(iii) $E = \bigcup_{i\in I} E_i$.

(9) A finite extension is normal if and only if it is a splitting field of a polynomial.

(10) If E/F is a normal extension, then $\mathrm{Aut}(E/F) \simeq \varprojlim_{E'} \mathrm{Aut}(E'/F)$ where E' runs through subfields of E such that E'/F is a finite normal extension.

(11) Suppose $F \subseteq E \subseteq K$, and E/F and K/F are normal extensions; then $\mathrm{Aut}(K/E)$ is a normal subgroup of $\mathrm{Aut}(K/F)$ and

$$\mathrm{Aut}(K/F)/\,\mathrm{Aut}(K/E) \simeq \mathrm{Aut}(E/F).$$

(12) Suppose E is a splitting field of $f(x)$ over F. Then $|\,\mathrm{Aut}(E/F)| \leq [E : F]$ and equality holds if and only if $f(x)$ is a separable polynomial; recall that we say a polynomial $f(x) \in F[x]$ is separable if all of its irreducible factors have distinct zeros in a splitting field of $f(x)$ over F.

(13) An algebraic extension E/F is called separable if for any $\alpha \in E$, $m_{\alpha,F}(x)$ is a separable polynomial.

(14) $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$ is not a separable extension.