Lecture 27: Proof of Krull's PIT

Monday, June 4, 2018 11:16 AM

We were in the middle of proof of Krull's PIT.

<u>Krull's PIT</u>. Suppose A is Noetherian, $a \notin A^{\times}$, and $\mathfrak{p}$ is a minimal prime ideal of $\langle a \rangle$. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

<u>Pf</u>. We have made a few reductions, and showed that in addition we can and will assume that A is a local integral domain with maximal ideal $\mathfrak{p}$. And assumed to the contrary that $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$ is an intermediate prime. We would like to show: $\operatorname{ht}(\mathfrak{p}_1) = 0$

$\operatorname{ht}(\mathfrak{p}_1) = 0 \iff \dim A_{\mathfrak{p}_1} = 0 \iff A_{\mathfrak{p}_1}$ is Artinian $\iff \exists\, n,\ \mathfrak{p}_1^n A_{\mathfrak{p}_1} = \mathfrak{p}_1^{n+1} A_{\mathfrak{p}_1}$.

Let $\mathfrak{p}_1^{(k)} := \mathfrak{p}_1^k A_{\mathfrak{p}_1} \cap A$. Since $\mathfrak{p}_1 A_{\mathfrak{p}_1}$ is maximal, $\mathfrak{p}_1^k A_{\mathfrak{p}_1}$ is $\mathfrak{p}_1 A_{\mathfrak{p}_1}$-primary. Hence $\mathfrak{p}_1^{(k)}$ is $\mathfrak{p}_1$-primary, and $\mathfrak{p}_1^{(1)} \supseteq \mathfrak{p}_1^{(2)} \supseteq \cdots$

Recall that $V(\langle a \rangle) = \{\mathfrak{p}\}$; and so $\dim A/\langle a \rangle = 0$, which implies $A/\langle a \rangle$ is Artinian. Hence $\exists n$, $\mathfrak{p}_1^{(n)} + \langle a \rangle = \mathfrak{p}_1^{(n+1)} + \langle a \rangle$.

So for any $x_n \in \mathfrak{p}_1^{(n)}$, $\exists\ x_{n+1} \in \mathfrak{p}_1^{(n+1)}$ and $v_n \in A$ s.t. $x_n = x_{n+1} + v_n a$. $(\dagger)$

$$\left.\begin{array}{l} \Rightarrow v_n a \in \mathfrak{p}_1^{(n)} \\ V(\langle a \rangle) = \{\mathfrak{p}\} \Rightarrow a \notin \mathfrak{p}_1 \end{array}\right\} \Rightarrow \left. \begin{array}{l} v_n \in \mathfrak{p}_1^{(n)} \\ (\dagger) \end{array}\right\} \Rightarrow \begin{array}{l} x_n \in \mathfrak{p}_1^{(n+1)} + \mathfrak{p}_1^{(n)} a \subseteq \mathfrak{p}_1^{(n)} \\ \Rightarrow \mathfrak{p}_1^{(n)} = \mathfrak{p}_1^{(n+1)} + a\, \mathfrak{p}_1^{(n)}. \end{array}$$

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As $a \in \mathfrak{p} = J(A)$, by Nakayama's lemma $\mathfrak{p}_1^{(n)} = \mathfrak{p}_1^{(n+1)}$; and so

$\mathfrak{p}_1^{(n)} A_{\mathfrak{p}_1} = \mathfrak{p}_1^{(n+1)} A_{\mathfrak{p}_1}$, which implies $\mathfrak{p}_1^{n} A_{\mathfrak{p}_1} = \mathfrak{p}_1^{n+1} A_{\mathfrak{p}_1}$; and claim

follows. ■

<u>Theorem</u>. Suppose A is Noetherian, $\mathfrak{p} \in \operatorname{Spec} A$, and $\operatorname{ht}(\mathfrak{p}) = d$.

Then $\exists\ \mathfrak{a} = \langle a_1, \ldots, a_d \rangle$ s.t. $\mathfrak{p}$ is a minimal prime in $V(\mathfrak{a})$.

<u>Pf.</u> Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d =: \mathfrak{p}$ be a chain of prime ideals.

Then $\operatorname{ht}(\mathfrak{p}_i) = i$ as $\operatorname{ht}(\mathfrak{p}_d) = \operatorname{ht}(\mathfrak{p}) = d$. We prove $\exists\ a_1, \ldots, a_d$ s.t.

(1) $\operatorname{ht}(\langle a_1, \ldots, a_k \rangle) = k$

(2) $\mathfrak{p}_k$ is a minimal prime in $V(\langle a_1, \ldots, a_k \rangle)$.

And we prove this by induction on d.

<u>Base of induction</u>. Since $\operatorname{ht} \mathfrak{p}_1 = 1$, $\mathfrak{p}_1 \not\subseteq \bigcup_{\operatorname{ht} \mathfrak{p}' = 0} \mathfrak{p}'$

(as otherwise $\mathfrak{p}_1 \subseteq \mathfrak{p}'$ for some $\mathfrak{p}'$ with $\operatorname{ht} \mathfrak{p}' = 0$) Let $a_1 \in \mathfrak{p}_1 \setminus \bigcup_{\operatorname{ht} \mathfrak{p}' = 0} \mathfrak{p}'$

Since a_1 is not in a minimal prime ideal, $\operatorname{ht}(\langle a_1 \rangle) \geq 1$; and by

Krull's PIT, we deduce $\operatorname{ht}(\langle a_1 \rangle) = 1$; and so $\mathfrak{p}_1$ is a minimal

prime in $V(\langle a_1 \rangle)$.

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<u>Induction Step.</u> Suppose $a_1, \ldots, a_{d-1}$ satisfy the mentioned conditions. Suppose $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_m$ are minimal prime in $V(\langle a_1, \ldots, a_{d-1}\rangle)$. By Krull's height theorem $\mathrm{ht}(\mathfrak{p}'_i) \leq d-1$ and by the induction hypothesis $\mathrm{ht}(\langle a_1, \ldots, a_{d-1}\rangle) = d-1$ which means $\min \mathrm{ht}(\mathfrak{p}'_i) = d-1$; and so $\mathrm{ht}\, \mathfrak{p}'_i = d-1$ for any i. Since $\mathrm{ht}\, \mathfrak{p}_d = d$, we deduce that $\mathfrak{p}_d \not\subseteq \bigcup_{i=1}^{m} \mathfrak{p}'_i$. Let $a_d \in \mathfrak{p}_d \setminus \bigcup_{i=1}^{m} \mathfrak{p}'_i$.

Suppose $\mathfrak{p}'$ is a minimal prime of $\langle a_1, \ldots, a_d\rangle$. Hence $\langle a_1, \ldots, a_{d-1}\rangle \subseteq \mathfrak{p}'$; and $\exists i,\ \mathfrak{p}'_i \subseteq \mathfrak{p}'$. As $a_d \in \mathfrak{p}' \setminus \mathfrak{p}'_i$, $\mathfrak{p}'_i \subsetneq \mathfrak{p}'$. Since $\mathrm{ht}\, \mathfrak{p}'_i = d-1$, $\mathrm{ht}\, \mathfrak{p}' \geq d$. And by Krull's HT $\mathrm{ht}\, \mathfrak{p}' \leq d$; and these imply $\mathrm{ht}\, \mathfrak{p}' = d$.

As $\mathfrak{p}_d \in V(\langle a_1, \ldots, a_d\rangle)$ and $\mathrm{ht}\, \mathfrak{p}_d = d$, we deduce that $\mathfrak{p}_d$ is a minimal prime in $V(\langle a_1, \ldots, a_d\rangle)$. ∎

Lecture 27: Dimension and number of generators

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<u>Theorem</u>. Suppose A is Noetherian, and $\mathrm{Max}\, A = \{\mathfrak{m}\}$. Then

$$\dim A = \min_{\mathfrak{q}:\ \mathfrak{m}\text{-primary}} d(\mathfrak{q})$$

where $d(\mathfrak{q})$ is the minimum number of generators of $\mathfrak{q}$.

<u>Pf.</u> If $\mathfrak{q}$ is $\mathfrak{m}$-primary, then $\sqrt{\mathfrak{q}} = \mathfrak{m}$; and $V(\mathfrak{q}) = \{\mathfrak{m}\}$.

And so by Krull's HT, $\mathrm{ht}(\mathfrak{m}) \le d(\mathfrak{q})$. Hence

$$\dim A = \mathrm{ht}(\mathfrak{m}) \le \min_{\mathfrak{q}:\ \mathfrak{m}\text{-primary}} d(\mathfrak{q}).$$

If $\mathrm{ht}(\mathfrak{m}) = d$, then by the previous theorem $\exists a_1, \ldots, a_d$ s.t.

$\mathfrak{m}$ is a minimal prime in $V(\langle a_1, \ldots, a_d\rangle)$. Hence

$V(\langle a_1, \ldots, a_d\rangle) = \{\mathfrak{m}\}$, which implies $\sqrt{\langle a_1, \ldots, a_d\rangle} = \mathfrak{m}$; and so

$\mathfrak{q}_0 := \langle a_1, \ldots, a_d\rangle$ is $\mathfrak{m}$-primary. Hence

$$\dim A = \mathrm{ht}\, \mathfrak{m} = d \ge d(\mathfrak{q}_0) \ge \min_{\mathfrak{q}:\ \mathfrak{m}\text{-primary}} d(\mathfrak{q});$$

and claim follows. ■

<u>Corollary</u>. Suppose A is Noetherian and $\mathrm{Max}\, A = \{\mathfrak{m}\}$. Then

$$\dim A \le \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2.$$

<u>Pf.</u> By Nakayama's lemma $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = d(\mathfrak{m})$; and claim follows. ■

Lecture 27: A few remarks

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This is an expanded version of a remark that I made during lecture.

- A local Noetherian ring A with a maximal ideal $\mathfrak{m}$ is called <u>regular</u> if $\dim A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

- For $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, the ring of poly. restricted to $X = X(f_1, \ldots, f_r)$ is isomorphic to $A := k[x_1, \ldots, x_n]/\sqrt{\langle f_1, \ldots, f_r \rangle}$.

Suppose $p \in X(f_1, \ldots, f_r)$; and let's consider all the rational functions that are defined at p and restrict them to X. Assuming that A is an integral domain, this ring can be naturally identified with $A_{\mathfrak{m}_p}$; and it is a local Noetherian ring. The tangent plane of X at p can be identified with the dual of $\tilde{\mathfrak{m}}_p/\tilde{\mathfrak{m}}_p^2$ where $\tilde{\mathfrak{m}}_p := \mathfrak{m}_p A_{\mathfrak{m}_p}$; and the regularity assumption $\dim A = \dim_{A/\mathfrak{m}_p} \tilde{\mathfrak{m}}_p/\tilde{\mathfrak{m}}_p^2$ is the same as saying that X does not have a singularity at p.