Lecture 25: Height of an ideal

Tuesday, May 29, 2018 12:57 AM

<u>Def</u>. Suppose A is a unital commutative ring.

For $\mathfrak{p} \in \operatorname{Spec} A$, $\operatorname{ht}(\mathfrak{p}) := \max\{n \in \mathbb{Z}^{\geq 0} \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}$,

for $\mathfrak{a} \underset{\neq}{\trianglelefteq} A$, $\operatorname{ht}(\mathfrak{a}) := \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{a} \subseteq \mathfrak{p}\} = \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in V(\mathfrak{a})\}$.

<u>Remark</u>. If $\mathfrak{a}$ is decomposable, then, for any $\mathfrak{p} \in V(\mathfrak{a})$, there is $\mathfrak{p}' \in \operatorname{Ass}(\mathfrak{a})$ s.t. $\mathfrak{p}' \subseteq \mathfrak{p}$. Hence

$$\operatorname{ht}(\mathfrak{a}) = \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})\}.$$

<u>Recall</u>. $\operatorname{Ass}(\mathfrak{a}) = \operatorname{Spec}(A) \cap \{\sqrt{(\mathfrak{a}:x)} \mid x \in A\}$.

(This is how we proved the 1^{st} uniqueness theorem.)

(Recall) <u>Pf</u>. Suppose $\mathfrak{a} = \bigcap_{i=1}^{m} \mathfrak{q}_i$ is a reduced primary decomposition, and $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Then

$(\mathfrak{a}:x) = \bigcap_{i=1}^{m} (\mathfrak{q}_i : x) = \bigcap_{x \notin \mathfrak{q}_i} (\mathfrak{q}_i : x)$; and so

$\sqrt{(\mathfrak{a}:x)} = \bigcap_{x \notin \mathfrak{q}_i} \sqrt{(\mathfrak{q}_i : x)} = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i$. (Recall $(\mathfrak{q}_i : x)$ is $\mathfrak{p}_i$-primary)

- If $\sqrt{(\mathfrak{a}:x)}$ is prime, then $\bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i = \mathfrak{p}$; this implies $\mathfrak{p} = \mathfrak{p}_i$ for some i.
- Since it is a reduced primary decomp., $\exists x_i \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$; this implies $\sqrt{(\mathfrak{a}:x_i)} = \mathfrak{p}_i$. ∎

Lecture 25: Improved description of associated primes

Friday, June 1, 2018 12:19 AM

For Noetherian rings this result can be improved:

<u>Proposition</u>. Suppose A is Noetherian. Then

$$\mathrm{Ass}(\mathfrak{a}) = \mathrm{Spec}(A) \cap \{(\mathfrak{a}:x) \mid x \in A\}.$$

<u>Pf.</u>
- If $(\mathfrak{a}:x)$ is prime, then $(\mathfrak{a}:x) = \sqrt{(\mathfrak{a}:x)} \in \mathrm{Spec}\, A$ and so $(\mathfrak{a}:x) \in \mathrm{Ass}(\mathfrak{a})$.
- Suppose $\mathfrak{a} = \bigcap_{i=1}^m \mathfrak{q}_i$ is a reduced primary decomposition, and $x_i \in \bigcap_{\substack{j=1\\ j\neq i}}^m \mathfrak{q}_j \setminus \mathfrak{q}_i$. Then $\mathfrak{p}_i = \sqrt{(\mathfrak{a}:x_i)}$. Since A is Noeth., $\exists\, n \in \mathbb{Z}^+$, $\mathfrak{p}_i^n \subseteq (\mathfrak{a}:x_i)$ and $\mathfrak{p}_i^{n-1} \not\subseteq (\mathfrak{a}:x_i) = (\mathfrak{q}_i:x_i)$.

If $n=1$, we are done.

If $n>1$, let $y \in \mathfrak{p}_i^{n-1} \setminus (\mathfrak{q}_i:x_i)$. Then $yx_i \in \bigcap_{\substack{j=1\\ j\neq i}} \mathfrak{q}_j \setminus \mathfrak{q}_i$; and so

1. $(\mathfrak{a}: yx_i) = (\mathfrak{q}_i : y x_i)$ is $\mathfrak{p}_i$-primary.

On the other hand, $\mathfrak{p}_i\, y x_i \subseteq \mathfrak{p}_i^n x_i \subseteq \mathfrak{a}$; this implies

2. $\mathfrak{p}_i \subseteq (\mathfrak{a}:yx_i)$. And so by 1 and 2 $\mathfrak{p}_i = (\mathfrak{a}:yx_i)$. ∎

<u>Cor.</u> A: Noetherian, $\mathrm{ht}(\langle a\rangle) = 0 \Rightarrow a \in D(A)$ (is a zero-div.)

Lecture 25: Height zero principal ideals

Friday, June 1, 2018 1:11 AM

Pf. $ht(\langle a\rangle)=0 \Rightarrow \exists \mathfrak{p}\in \mathrm{Ass}(\langle a\rangle),\ ht(\mathfrak{p})=0$

$\Rightarrow \mathfrak{p}\in \mathrm{Ass}(\langle a\rangle)$ and $\mathfrak{p}$ minimal element of $\mathrm{Spec}\, A$.

$\Rightarrow \mathfrak{p}\in \mathrm{Ass}(\langle a\rangle)\cap \mathrm{Ass}(0)$.

$\Rightarrow \exists\, x_1, x_2 \in A\setminus\{0\},\ \mathfrak{p}=(\langle a\rangle : x_1)=(0:x_2)$

$\Rightarrow x_1\mathfrak{p}=\langle a\rangle$ and $\mathfrak{p}x_2=0$.

$\Rightarrow \left.\begin{array}{l} a x_2 \in x_1x_2\mathfrak{p}=0 \\ x_2\neq 0\end{array}\right\} \Rightarrow a\in D(A).$ ∎

Converse is not correct.

$A=k[x,y]/\langle x^2,xy\rangle$; $\cdot\ \langle x^2,xy\rangle=\langle x\rangle\cap\langle x,y\rangle^2$

$\cdot\ \mathrm{Ass}(\langle x^2,xy\rangle)=\{\langle x\rangle,\langle x,y\rangle\}$.

$\Rightarrow D(A)=\bigcup_{\mathfrak{p}\in \mathrm{Ass}(0)}\mathfrak{p}=\langle \bar{x},\bar{y}\rangle$ and

$ht(\langle \bar{y}\rangle)=ht(\langle \bar{x},\bar{y}\rangle)=1$.

Krull's Principal Ideal Theorem.

$\left.\begin{array}{l} A: \text{Noetherian},\ a\notin U(A), \\ \mathfrak{p}: \text{minimal prime that contains } a\end{array}\right\} \Rightarrow ht(\mathfrak{p})\le 1.$

Lecture 25: Krull's height theorem

Friday, June 1, 2018 8:14 AM

Before we prove Krull's Principal Ideal Theorem, we prove some of its consequences: Krull's Height Theorem.

A: Noetherian; $\mathrm{ht}(\langle a_1, \dots, a_n\rangle) \leq n$ if $\langle a_1, \dots, a_n\rangle$ is proper. Moreover, if $\mathfrak{p}$ is minimal in $V(\mathfrak{a})$, then $\mathrm{ht}(\mathfrak{p}) \leq n$.

Pf. We proceed by induction on n. The base of induction follows from Krull's principal ideal theorem and the previous corollary.

The induction step. Suppose $\mathfrak{p} \in \mathrm{Ass}(\underbrace{\langle a_1, \dots, a_n\rangle}_{\mathfrak{a}})$ is a minimal prime that contains $\mathfrak{a}$. Let $\mathfrak{p}' \subsetneq \mathfrak{p}$ be a prime ideal and suppose there is no prime ideal between $\mathfrak{p}'$ and $\mathfrak{p}$. Why is there such $\mathfrak{p}'$? Otherwise we get an infinite chain of prime ideals, which is not possible as A is Noetherian.

Notice that $\mathrm{ht}(\mathfrak{p}) = \max\{\mathrm{ht}(\mathfrak{p}') \mid \mathfrak{p}' \subsetneq \mathfrak{p}\} + 1$, and $\mathrm{ht}(\mathfrak{a}) = \min\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(\mathfrak{a}) \text{ minimal}\}$. Hence if we show $\mathfrak{p}'$ is a minimal element of $\mathrm{Ass}(\mathfrak{a}')$ for some $\mathfrak{a}'$ that is generated by n-1 elements, by the induction hypothesis claim follows.