Lecture 16: Valuation rings

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We treat valuation rings as auxiliary tools.

Def Suppose A is an integral domain and F is its field of fractions.

We say A is a <u>valuation ring</u> if $\forall a\in F$, either $a\in A$ or $a^{-1}\in A$.

Lemma. Suppose A is a valuation ring with field of fractions F

(a) A is a local ring.

(b) If $A\subseteq A'\subseteq F$ is a subring, then A' is a valuation ring.

(c) A is integrally closed.

Pf. (a). Let $\mathfrak{m}:=A\setminus A^\times$. $\forall a\in A$, $b\in\mathfrak{m}$, $b\notin A^\times$ implies $ab\notin A^\times$

$\Rightarrow ab\in\mathfrak{m}$.

. $b_1,b_2\in\mathfrak{m}\setminus\{0\} \Rightarrow b_1/b_2\in A$ or $b_2/b_1\in A$

If $b_1/b_2\in A \Rightarrow b_1+b_2=\underbrace{b_2}_{\text{in } \mathfrak{m}}\underbrace{(b_1/b_2+1)}_{\text{in } A}\in\mathfrak{m}$.

If $b_2/b_1\in A\Rightarrow b_1+b_2=b_1(b_2/b_1+1)\in\mathfrak{m}$.

(b) is clear. (c) Suppose $f\in F$ is integral over A and $f\notin A$.

Then $f^{-1}\in A$ and $f^n+a_{n-1}f^{n-1}+\cdots+a_0=0$ for some $a_i\in A$. And so

$f=-(a_{n-1}+a_{n-2}f^{-1}+\cdots+a_0f^{-(n-1)})\in A$, which is a contradiction. ∎

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The following is the key technical result:

<u>Theorem</u>. Let Ω be an algebraically closed field, A_0 an integral domain, $\phi_0 : A_0 \to \Omega$ a ring homomorphism. Let

$$\Sigma := \left\{ (A', \phi') \;\middle|\; \begin{array}{ll} \cdot\ A_0 \subseteq A' \subseteq F & \text{subring} \\ \cdot\ \phi' : A' \to \Omega & \text{ring hom.} \\ \cdot\ \phi'|_{A_0} = \phi_0 & \end{array} \right\}$$

where F is a field. We say $(A_1, \phi_1) \preccurlyeq (A_2, \phi_2)$ if $A_1 \subseteq A_2$ and $\phi_2|_{A} = \phi_1$. Then Σ has a maximal element. And if (B, θ) is a maximal element of Σ, then B is a valuation ring whose field of fractions is F and $\ker \theta$ is its unique maximal ideal.

<u>Pf.</u> By Zorn's lemma, it is enough to show any chain $\{(A_i, \phi_i)\}_{i \in I}$ has an upper bound. Notice that $A := \bigcup_{i \in I} A_i$ is a subring of F that contains A_0. And

$\phi : A \to \Omega$, $\phi(a) = \phi_i(a)$ if $a \in A_i$ is a well-defined ring hom. And so $(A, \phi) \in \Sigma$ is an upper bound for $\{(A_i, \phi_i)\}_{i \in I}$.

Therefore by Zorn's lemma Σ has a maximal element.

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<u>Step 1.</u> B is a local ring and $\ker\theta$ is its maximal ideal.

<u>Pf of step 1.</u> Notice that $B/_{\ker\theta} \hookrightarrow \Omega$; and so $\ker\theta$ is a prime ideal. Since $B \subseteq F$, $B \hookrightarrow B_{\ker\theta} \hookrightarrow F$.

Since $\theta(B\setminus\ker\theta) \neq 0$, $\exists\ \hat{\theta}: B_{\ker\theta} \to \Omega$ s.t. $\hat{\theta}|_B = \theta$.

$\Rightarrow (B,\theta) \preccurlyeq (B_{\ker\theta}, \hat{\theta})$. By maximality of (B,θ) we have $B = B_{\ker\theta}$ and claim follows.

<u>Step 2.</u> $\forall\ \alpha \in F$, either $\mathfrak{m}[\alpha] \neq B[\alpha]$ or $\mathfrak{m}[\alpha^{-1}] \neq B[\alpha]$

where $\mathfrak{m} := \ker\theta$.

<u>Pf of step 2.</u> Suppose to the contrary that $1 \in \mathfrak{m}[\alpha] \cap \mathfrak{m}[\alpha^{-1}]$ and let m,n be the smallest positive integers s.t.

$(*)$ $\quad 1 = a_0 + a_1\alpha + \dots + a_n\alpha^n$ and $1 = a'_0 + a'_1\alpha^{-1} + \dots + a'_m\alpha^{-m}$

for some $a_i, a'_j \in \mathfrak{m}$. And suppose $n \geq m$.

Since $1 - a_0 \in 1 + \mathfrak{m} \subseteq B^\times$, $\exists\ a''_i \in \mathfrak{m}$ s.t.

$1 = a''_1\alpha^{-1} + \dots + a''_m\alpha^{-m} \Rightarrow \alpha^m \in \mathfrak{m} + \mathfrak{m}\alpha + \dots + \mathfrak{m}\alpha^{m-1}$

$\Rightarrow \alpha^n \in \mathfrak{m} + \dots + \mathfrak{m}\alpha^{n-1} \overset{(*)}{\Rightarrow} 1 \in \mathfrak{m} + \dots + \mathfrak{m}\alpha^{n-1}$ which is a

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contradiction.

<u>Step 3</u>. $\forall \alpha \in F$, either $\alpha \in B$ or $\alpha^{-1} \in B$.

<u>Pf.</u> Using Step 2, w.l.o.g. suppose $\mathfrak{m}[\alpha] \neq B[\alpha]$. So $\exists$ a maximal ideal $\mathfrak{m}'$ of $B[\alpha]$ that contains $\mathfrak{m}[\alpha]$. Hence

$$\left.\begin{array}{l} \mathfrak{m}' \cap B \in \operatorname{Spec} B \\ \quad\ \ \rotatebox{90}{$\subseteq$} \\ \quad\ \ \mathfrak{m} \end{array}\right\} \Rightarrow \mathfrak{m}' \cap B = \mathfrak{m}.$$ Therefore

$$\begin{tikzcd} B/\mathfrak{m} \arrow[r, hook] \arrow[d, "\overline{\theta}"'] & B[\alpha]/\mathfrak{m}' \arrow[dl, dashed, "\overline{\theta}'"', "?"] \\ \Omega & \end{tikzcd} \qquad (*)$$

is a field extension $\Rightarrow \overline{\alpha}$ is alg. over $B/\mathfrak{m}$

Since Ω is algebraically closed, $\exists\, \overline{\theta}'$ s.t. $(*)$ holds.

Let $\theta' : B[\alpha] \to \Omega$, $\theta'(\beta) := \overline{\theta}'(\beta)$. Then

$(B, \theta) \preccurlyeq (B[\alpha], \theta')$; and so $B = B[\alpha]$, and $\alpha \in B$. $\blacksquare$