Lecture 11: Integral morphisms are closed

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At the end of the previous lecture we proved:

<u>Theorem</u>. Suppose $f: A \hookrightarrow B$ is integral. Then $f^*: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is onto.

<u>Pf</u>: For $\mathfrak{p} \in \operatorname{Spec} A$, let $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$. Then $A_{\mathfrak{p}} \xhookrightarrow{f_{\mathfrak{p}}} S_{\mathfrak{p}}^{-1} B$ is integral. And so $f_{\mathfrak{p}}^*(\operatorname{Max} S_{\mathfrak{p}}^{-1} B) \subseteq \operatorname{Max} A_{\mathfrak{p}} = \{S_{\mathfrak{p}}^{-1}\mathfrak{p}\}$. Hence $\exists\, \mathfrak{q} \in \operatorname{Spec} B$ s.t. $\mathfrak{q} \cap S_{\mathfrak{p}} = \emptyset$ and $S_{\mathfrak{p}}^{-1}\mathfrak{q} \cap S_{\mathfrak{p}}^{-1} A = S_{\mathfrak{p}}^{-1}\mathfrak{p}$.

Therefore $\mathfrak{q} \cap A \subseteq \mathfrak{p}$ and $\mathfrak{p} \subseteq \mathfrak{q}$. Thus $f^*(\mathfrak{q}) = \mathfrak{p}$. ■

<u>Corollary</u>. Suppose $f: A \hookrightarrow B$ is integral. Then $f^*(\operatorname{Max} B) = \operatorname{Max} A$.

<u>Pf</u>. We have already proved that $f^*(\operatorname{Max} B) \subseteq \operatorname{Max} A$, and $(f^*)^{-1}(\operatorname{Max} A) \subseteq \operatorname{Max} B$. So claim follows from surjectivity of f^*. ■

<u>Corollary</u>. Suppose $A \xhookrightarrow{f} B$ is integral. Then $f^*: \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map.

<u>Pf</u>. <u>Claim</u>. $f^*(V(\mathfrak{b})) = V(\mathfrak{b}^c)$; let $\bar{f}: A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$. Then $\bar{f}$ is integral and so $\bar{f}^*$ is onto.

On the other hand, we have

$$\begin{tikzcd} \operatorname{Spec} B/\mathfrak{b} \arrow[r] \arrow[d, "\bar{f}^*"'] \arrow[dr, phantom, "\circlearrowleft"] & V(\mathfrak{b}) \arrow[d, "f^*"] \\ \operatorname{Spec} A/\mathfrak{b}^c \arrow[r] & V(\mathfrak{b}^c) \end{tikzcd}$$

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Lecture 11: Going-Up theorem and fibers

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<u>Theorem</u> (Going-Up theorem) Suppose $f: A \hookrightarrow B$ is integral, $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ is a chain in Spec A, and $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ is a chain is Spec B such that $f^*(\mathfrak{q}_i) = \mathfrak{p}_i$. Then $\exists\, \mathfrak{q}_{m+1} \subsetneq \cdots \subsetneq \mathfrak{q}_n$ in Spec B such that $f^*(\mathfrak{q}_i) = \mathfrak{p}_i$.

(Going-Up)

<u>Pf.</u> We proceed by induction on m. The case $m=-1$ is a conseq. of surjectivity of f^*. To prove the induction step, it is enough to prove the case of $m=0$: $f^*(\mathfrak{q}_0) = \mathfrak{p}_0$ and $\mathfrak{p}_1 \in V(\mathfrak{p}_0)$.

Then by the previous corollary $f^*(V(\mathfrak{q}_0)) = V(\mathfrak{p}_0)$, and so $\exists\, \mathfrak{q}_1 \in V(\mathfrak{q}_0)$ s.t. $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$; and claim follows. ■

Next we show dimension of any fiber $(f^*)^{-1}(\mathfrak{p})$ is zero:

<u>Theorem</u>. Suppose $f: A \hookrightarrow B$ is integral, and for $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \in$ Spec B, $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2) = \mathfrak{p}$. Then $\mathfrak{q}_1 = \mathfrak{q}_2$.

<u>Pf.</u> Since $A \hookrightarrow B$ is integral, $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}_1$ is integral. So w.l.o.g. we can and will assume $\mathfrak{p} = \mathfrak{q}_1 = 0$; in particular A and B are

which implies $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$. ■

Lecture 11: Integral extension and dimension

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__Theorem__. Suppose $A \overset{f}{\hookrightarrow} B$ is integral. Then $\dim A = \dim B$.

__Pf.__ • Suppose $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ is a chain in Spec B. Since fibers have dimension zero, $f^*(\mathfrak{q}_0) \subsetneq \cdots \subsetneq f^*(\mathfrak{q}_m)$. Therefore $\dim B \le \dim A$

• Suppose $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ is a chain in Spec A. Then by Going-Up theorem $\exists\ \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ in Spec B. And so $\dim A \le \dim B$; and claim follows. ∎

Next we will show under extra assumption an integral morphism is open as well (as closed). We need some auxiliary results.

__Def.__ Suppose B/A is a ring extension, and $\mathfrak{a} \trianglelefteq A$. Then $b \in B$ is called __integral over $\mathfrak{a}$__ if $\exists\ a_i \in \mathfrak{a}$ s.t.

$$b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0.$$

__Lemma__. Suppose B/A is a ring extension, and $\mathfrak{a} \trianglelefteq A$. Let C be the integral closure of A in B. Then $b \in B$ is integral over $\mathfrak{a}$ if and only if $b \in \sqrt{\mathfrak{a}^e}$ where $\mathfrak{a}^e$ is the extension of $\mathfrak{a}$ in C.

__Pf.__ ($\Rightarrow$) Since b is integral over $\mathfrak{a}$, $b \in C$ and $b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$

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for some $a_0, \dots, a_{n-1} \in \mathfrak{a}$. Hence

$$b^n = -a_{n-1} b^{n-1} - \dots - a_1 b - a_0 \in \mathfrak{a}^e;$$

and so $b \in \sqrt{\mathfrak{a}^e}$.

($\Leftarrow$) Suppose $b \in \sqrt{\mathfrak{a}^e}$. So $b^n = a_1 c_1 + \dots + a_m c_m$ for some $a_i \in \mathfrak{a}$ and $c_i \in C$. Let $M := A[c_1, \dots, c_m]$. Since c_i's are integral over A, M is a f.g. A-module.

$$\left.\begin{aligned} b^n M = \sum_{i=1}^m a_i c_i M \subseteq \mathfrak{a} M \;;\; \ell_{b^n} \in \mathrm{End}_A(M) \end{aligned}\right\} \Rightarrow$$

$$\exists\, a'_0, \dots, a'_{s-1} \in \mathfrak{a},\ \left((b^n)^s + a'_{s-1}(b^n)^{s-1} + \dots + a'_0\right) M = 0.$$

As $1 \in M$, $(b^n)^s + a'_{s-1}(b^n)^{s-1} + \dots + a'_0 = 0$ and $a'_i \in \mathfrak{a}$.

Therefore b is integral over $\mathfrak{a}$. ∎

<u>Corollary</u>. If b and b' are integral over $\mathfrak{a}$, then $b \pm b'$ and bb' are integral over $\mathfrak{a}$.