Lecture 08: Krull dimension one and primary decomposition

Tuesday, April 17, 2018 10:53 PM

In the previous lecture we proved the 2^{nd} uniqueness theorem which implies:

If $\mathfrak{a}$ is decomposable, for any minimal element $\mathfrak{p}$ of $\mathrm{Ass}(\mathfrak{a})$ $\mathfrak{p}$-factors of reduced primary decompositions of $\mathfrak{a}$ are the same.

We will see an application of this for Krull dimension 1 integral domains.

<u>Def.</u> $\dim A := \sup\{n \in \mathbb{Z}^{\geq 0} \mid \exists \mathfrak{p}_i \in \mathrm{Spec}(A),\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.

<u>Ex.</u> k: field $\Rightarrow \dim k = 0$.

<u>Ex.</u> Suppose A is an integral domain. Then $\dim A = 1$ if and only if $0 \notin \mathrm{Max}(A)$ and $\mathrm{Spec}(A) = \{0\} \cup \mathrm{Max}(A)$; in particular $\dim(A) = 1$ if A is a PID and not a field.

<u>Pf.</u> ($\Rightarrow$) If $0 \in \mathrm{Max}(A)$, then $\dim A = 0$; that is a contrad.

Suppose to the contrary $\exists\, \mathfrak{p} \in \mathrm{Spec}(A) \setminus (\{0\} \cup \mathrm{Max}(A))$. Then $\exists\, \mathfrak{m} \in \mathrm{Max}(A)$ s.t. $\mathfrak{p} \subsetneq \mathfrak{m} \Rightarrow 0 \subsetneq \mathfrak{p} \subsetneq \mathfrak{m} \Rightarrow \dim A \geq 2$; this is a contradiction. ($\Leftarrow$) Since $0 \notin \mathrm{Max}(A)$, for $\mathfrak{m} \in \mathrm{Max}(A)$ we get $0 \subsetneq \mathfrak{m}$. And so $\dim A \geq 1$. If $\exists\, 0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ for some $\mathfrak{p}_i \in \mathrm{Spec}(A)$, then $\mathfrak{p}_2 \in \mathrm{Spec}(A) \setminus (\{0\} \cup \mathrm{Max}(A))$ which is a contr.

Lecture 08: Dimension one integral domains

Monday, April 16, 2018 12:25 AM

We have seen that, if A is a PID, then $\mathrm{Spec}(A)=\{0\}\cup \mathrm{Max}(A)$.

And claim follows.

<u>Proposition</u>. Suppose D is an integral domain and $\dim D=1$. Suppose $\mathfrak{a}\trianglelefteq D$ is decomposable. Then $\mathfrak{a}$ has a <u>unique</u> <u>reduced primary decomposition.</u>

<u>Pf.</u> • If $\mathfrak{a}=0$, then $\mathfrak{a}$ is prime; and so by the 1^{st} uniqueness theorem we are done.

• If $\mathfrak{a}\neq 0$, then $\mathrm{Ass}(\mathfrak{a})\subseteq \mathrm{Max}(A)$; and so any element of $\mathrm{Ass}(\mathfrak{a})$ is minimal in $\mathrm{Ass}(\mathfrak{a})$. Hence by the 2^{nd} uniqueness theorem claim follows. ■

<u>Corollary</u>. Suppose D is an integral domain and $\dim D=1$. Suppose $0\neq\mathfrak{a}\ntrianglelefteq D$ is decomposable. Then there are unique primary ideals (up to permutation) $\mathfrak{q}_1,\ldots,\mathfrak{q}_n$ s.t.

$$\mathfrak{a}=\prod_{i=1}^{n}\mathfrak{q}_i \quad \text{, and} \quad \sqrt{\mathfrak{q}_i}\neq\sqrt{\mathfrak{q}_j} \text{ if } i\neq j.$$

Pf. Let $\mathfrak{a}=\bigcap_{i=1}^{n}\mathfrak{q}_i$ be a reduced primary decomposition.

Then $\sqrt{\mathfrak{q}_i} \in \mathrm{Max}(A)$ for any i, and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ if $i \neq j$. Hence $\sqrt{\mathfrak{q}_i}, \sqrt{\mathfrak{q}_j}$ are coprime. Therefore $\mathfrak{q}_i$ and $\mathfrak{q}_j$ are coprime. And so $\bigcap_{i=1}^{n} \mathfrak{q}_i = \prod_{i=1}^{n} \mathfrak{q}_i$.

Suppose $\mathfrak{a} = \prod_{i=1}^{m} \mathfrak{q}_i'$ such that $\mathfrak{q}_i'$ is $\mathfrak{p}_i'$-primary and $\mathfrak{p}_i' \neq \mathfrak{p}_j'$ if $i \neq j$. Then again $\mathfrak{q}_i'$ and $\mathfrak{q}_j'$ are coprime, and so $\mathfrak{a} = \bigcap_{i=1}^{m} \mathfrak{q}_i'$ is a primary decomposition. To get the uniqueness it is enough to show this decomposition is <u>reduced</u>. If $\mathfrak{q}_i' \supseteq \bigcap_{\substack{j=1 \\ j\neq i}}^{m} \mathfrak{q}_j'$ then $\mathfrak{p}_i' \mid \prod_{\substack{j=1 \\ j\neq i}}^{m} \mathfrak{q}_j'$; and so $\mathfrak{p}_i' \mid \mathfrak{q}_j'$ for some $j \neq i$; this implies $\mathfrak{p}_j' \subsetneq \mathfrak{p}_i'$ which contradicts $\mathfrak{p}_j' \in \mathrm{Max}(A)$. ∎

Next we will show that a reduced primary decomposition exists if A is Noetherian. Similar to the case of working with elements, we will define <u>irreducible ideals</u> and work with them.

<u>Def</u>. $\mathfrak{a} \underset{\neq}{\trianglelefteq} A$ is called irreducible if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, $\mathfrak{b} \underset{\neq}{\trianglelefteq} A$, $\mathfrak{c} \underset{\neq}{\trianglelefteq} A$ imply either $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

Lecture 08: Existence of a primary decomposition

Monday, April 16, 2018 12:38 AM

Next proposition gives us the connection between irreducible and primary ideals.

Proposition. An irreducible ideal is primary if A is Noetherian.

Pf. Suppose $\mathfrak{a} \underset{\neq}{\trianglelefteq} A$ is irreducible. Replacing A with $A/\mathfrak{a}$, we can assume $\underline{0}$ is irreducible, and we have to show any zero-div. is nilpotent. So suppose $xy=0$ and $y=0$. Now consider $(0:x) \subseteq (0:x^2) \subseteq \cdots$. Since A is Noetherian, $\exists\, n \in \mathbb{Z}^+$ s.t. $(0:x^n)=(0:x^{n+1})$.

Claim. $\langle x^n \rangle \cap \langle y \rangle = 0$

Pf. Suppose $z \in \langle x^n \rangle \cap \langle y \rangle$. Then $z = x^n a = y a'$. Then $xz = xya' = 0$; and so $x^{n+1} a = xz = 0$ which implies $a \in (0:x^{n+1}) = (0:x^n)$; therefore $z = x^n a = 0$. $\square$

Since 0 is irreducible and $y \neq 0$, by the above claim $x^n = 0$ and so x is nilpotent and claim follows. $\blacksquare$

Lecture 08: Existence of a primary decomposition

Monday, April 16, 2018 8:41 AM

Theorem. In a Noetherian ring, any proper ideal has a primary decomposi.

Pf. By the previous proposition, it is enough to show any proper ideal can be written as an intersection of finitely many irreducible ideals. Let

$$\Sigma := \{ \mathfrak{a} \underset{\neq}{\trianglelefteq} A \mid \mathfrak{a} \text{ cannot be written as an intersection of finitely many irreducible ideals} \}.$$

If $\Sigma \neq \emptyset$, then it has a maximal element as A is Noetherian.

Say $\mathfrak{a} \triangleleft \Sigma$ is a maximal element of Σ. Then $\mathfrak{a}$ is NOT irred.

And so $\exists\, \mathfrak{b}, \mathfrak{c} \underset{\neq}{\trianglelefteq} A$ s.t. $\mathfrak{b} \underset{\neq}{\supseteq} \mathfrak{a}$ $^{(1)}$ and $\mathfrak{c} \underset{\neq}{\supseteq} \mathfrak{a}$ $^{(2)}$ and $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ $^{(3)}$.

As $\mathfrak{a}$ is maximal in Σ, by (1) and (2) $\mathfrak{b}, \mathfrak{c} \notin \Sigma$. Therefore

$\exists$ irreducible ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ and $\mathfrak{q}_{m+1}, \ldots, \mathfrak{q}_n$ s.t.

$\mathfrak{b} = \bigcap_{i=1}^{m} \mathfrak{q}_i$ and $\mathfrak{c} = \bigcap_{i=m+1}^{n} \mathfrak{q}_i$. Therefore $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ can be written as an intersection of finitely many irreducible ideals, which is a contradiction. ■

Corollary. If A is Noetherian, then $\operatorname{Spec}(A)$ has only finitely many minimal elements $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$; in particular $\operatorname{Spec}(A) = \overline{\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}}$.

Lecture 08: Minimal primes

Tuesday, April 17, 2018 11:46 PM

Pf. Since A is Noetherian, 0 is decomposable. So

$$\{\text{minimal elements of } \mathrm{Spec}(A)\} = \{\text{minimal elements of } \mathrm{Ass}(0)\}$$
$$= \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

Suppose $\overline{\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}} = V(\mathfrak{a})$ and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then

$\exists i, \mathfrak{p}_i \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \mid \mathfrak{p}_i \mid \mathfrak{a} \Rightarrow \mathfrak{p} \in V(\mathfrak{a})$; and so

$\overline{\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}} = \mathrm{Spec}(A)$. ∎