Lecture 02: Union of prime ideals

Monday, April 2, 2018 8:53 AM

In the previous lecture we mentioned that union of ideals is often far from being an ideal and the following is a good indication of this fact.

<u>Proposition.</u> Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathrm{Spec}(A)$. If $\mathfrak{a} \trianglelefteq A$ and $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$, then $\exists\, i$, $\mathfrak{a} \subseteq \mathfrak{p}_i$.

<u>Pf.</u> We proceed by induction on n. Suppose $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ $(\forall i)$. Then by the induction hypothesis, for any i, $\exists\, x_i \in \mathfrak{a} \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n+1} \mathfrak{p}_j$.

And so $\forall i$, $x_i \in (\mathfrak{a} \cap \mathfrak{p}_i) \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{n+1} \mathfrak{p}_j$. Let

$y = x_1 + x_2 \cdots x_n$. Then $y \in \mathfrak{a}$ implies $y \in \mathfrak{p}_i$ for some i.

<u>Case 1.</u> $i=1$. $x_1 + x_2 \cdots x_{n+1} \in \mathfrak{p}_1 \Rightarrow x_2 \cdots x_{n+1} \in \mathfrak{p}_1$

$\Rightarrow \exists\, 2 \leq j \leq n+1$, $x_j \in \mathfrak{p}_1$ which is a contradiction.

<u>Case 2.</u> $2 \leq i \leq n+1$. $x_1 + \underbrace{x_2 \cdots x_{n+1}}_{\text{in } \mathfrak{p}_i} \in \mathfrak{p}_i \Rightarrow x_1 \in \mathfrak{p}_i$ which is a contradiction. ∎

In your homework, you will see a generalization of this proposition, where the primeness assumption is removed. (This is due to McCoy.)

Lecture 02: Prime divisors of an ideal

Tuesday, April 3, 2018   11:47 PM

Def. We say $\mathfrak{p} \in \operatorname{Spec}(A)$ is a prime divisor of $\mathfrak{a}$ if $\mathfrak{a} \subseteq \mathfrak{p}$,

and let $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

- For $\mathfrak{a}, \mathfrak{b} \lhd A$, we say $\mathfrak{b} \mid \mathfrak{a}$ if $\mathfrak{a} \subseteq \mathfrak{b}$.

Lemma. • $\mathfrak{a} \mid \mathfrak{b}$ and $\mathfrak{b} \mid \mathfrak{c} \Rightarrow \mathfrak{a} \mid \mathfrak{c}$

- $\mathfrak{a} \mid \mathfrak{b} \Rightarrow V(\mathfrak{a}) \subseteq V(\mathfrak{b})$.

(Clear).

Proposition. (0) $V((1)) = \emptyset$, $V(0) = \operatorname{Spec}(A)$.

(1) $V(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$.

(2) $V(\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_n) = \bigcup_{i=1}^{n} V(\mathfrak{a}_i)$.

Pf. (0) Since any prime ideal is proper and contains 0,

we get (0).

(1) $\sum_{i \in I} \mathfrak{a}_i \mid \mathfrak{a}_j \Rightarrow V(\sum_{i \in I} \mathfrak{a}_i) \subseteq V(\mathfrak{a}_j)$

$$\Rightarrow V(\sum_{i \in I} \mathfrak{a}_i) \subseteq \bigcap_{j \in I} V(\mathfrak{a}_j).$$

$$\mathfrak{p} \in \bigcap_{j \in I} V(\mathfrak{a}_j) \Rightarrow \forall j,\ \mathfrak{a}_j \subseteq \mathfrak{p} \Rightarrow \sum_{i \in I} \mathfrak{a}_i \subseteq \mathfrak{p}$$

$$\Rightarrow \mathfrak{p} \in V(\sum_{i \in I} \mathfrak{a}_i).$$

Lecture 02: Zariski topology

Wednesday, April 4, 2018 10:12 AM

$$\mathfrak{a}_i \mid \mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_n \Rightarrow V(\mathfrak{a}_i) \subseteq V(\mathfrak{a}_1 \cdots \mathfrak{a}_n)$$

$$\Rightarrow \bigcup_{i=1}^{n} V(\mathfrak{a}_i) \subseteq V(\mathfrak{a}_1 \cdots \mathfrak{a}_n)$$

Suppose $\mathfrak{p} \in V(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \setminus \bigcup_{i=1}^{n} V(\mathfrak{a}_i)$. So $\mathfrak{a}_1 \cdots \mathfrak{a}_n \subseteq \mathfrak{p}$

and $\mathfrak{a}_i \not\subseteq \mathfrak{p}$. So $\exists\, x_i \in \mathfrak{a}_i \setminus \mathfrak{p}$. Hence $x_1 x_2 \cdots x_n \in \mathfrak{p}$.

As $\mathfrak{p}$ is prime, $\exists\, i$, $x_i \in \mathfrak{p}$ which is a contradiction. ■

<u>Def.</u> Consider $\{V(\mathfrak{a})\}_{\mathfrak{a} \triangleleft A}$ as the set of closed subsets of $\operatorname{Spec}(A)$.

Previous Proposition shows that this defines a topology on $\operatorname{Spec}(A)$.

This is called the Zariski-topology on $\operatorname{Spec}(A)$.

Suppose $f: A \to B$ is a ring homomorphism. Then for any

ideal $\mathfrak{b}$ of B, $f^{-1}(\mathfrak{b})$ is an ideal of A and it is called

the <u>contraction of $\mathfrak{b}$</u>, it is denoted by $\mathfrak{b}^c$.

<u>Lemma.</u> $\mathfrak{b}^c \triangleleft A$ and $A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$.

<u>Pf.</u> $A \xrightarrow{f} B \xrightarrow{\pi} B/\mathfrak{b}$, with $A \xrightarrow{\pi \circ f} B/\mathfrak{b}$; then $\ker(\pi \circ f) = f^{-1}(\mathfrak{b}) = \mathfrak{b}^c$.

And so $\mathfrak{b}^c \triangleleft A$ and by the 1^{st} isomorphism theorem, $A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$. ■

Lecture 02: Contraction

Wednesday, April 4, 2018 12:06 AM

Lemma. Suppose $f: A \to B$ is a ring homomorphism. Then

$f^*: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $f^*(\mathfrak{p}) := \mathfrak{p}^c$ is a well-defined

continuous map (w.r.t. the Zariski-topology).

Pf. By the previous lemma, $A/f^*(\mathfrak{p}) \hookrightarrow B/\mathfrak{p}$.

As $B/\mathfrak{p}$ is an integral domain, $A/f^*(\mathfrak{p})$ is an integral

domain. (Since $f(1)=1$, $f^*(\mathfrak{p}) \neq A$). And so

$f^*(\mathfrak{p}) \in \mathrm{Spec}(A)$.

- $(f^*)^{-1}(V(\mathfrak{a})) \ni \mathfrak{p} \iff f^*(\mathfrak{p}) \in V(\mathfrak{a})$

$$\iff \mathfrak{a} \subseteq f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$$

$$\iff f(\mathfrak{a}) \subseteq \mathfrak{p}$$

$$\iff \langle f(\mathfrak{a}) \rangle \subseteq \mathfrak{p}$$

this is called the extension of $\mathfrak{a}$ w.r.t. f and it is denoted by $\mathfrak{a}^e$.

$$\iff \mathfrak{p} \in V(\mathfrak{a}^e)$$

preimage of a closed set is a closed set.

And $(f^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$. ∎

Lecture 02: Closed immersion

Wednesday, April 4, 2018 10:13 AM

Lemma. Suppose $\mathfrak{a} \trianglelefteq A$ and $\pi : A \to A/\mathfrak{a}$ is the natural quotient map. Then π^* induces a bijection from $\mathrm{Spec}(A/\mathfrak{a})$ to $V(\mathfrak{a})$.

Pf. Let $\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{a})$. Then $\mathfrak{p}^c = \pi^{-1}(\mathfrak{p}) \supseteq \pi^{-1}(0) = \mathfrak{a}$ and so $\pi^*(\mathrm{Spec}(A/\mathfrak{a})) \subseteq V(\mathfrak{a})$.

. Suppose $\tilde{\mathfrak{p}} \in V(\mathfrak{a})$; then let $\mathfrak{p} := \tilde{\mathfrak{p}}/\mathfrak{a}$. By isomor. theorems, $A/\tilde{\mathfrak{p}} \simeq \dfrac{A/\mathfrak{a}}{\tilde{\mathfrak{p}}/\mathfrak{a}}$; and so $\dfrac{A/\mathfrak{a}}{\mathfrak{p}}$ is an integral domain. Therefore $\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{a})$; and clearly $\pi^*(\mathfrak{p}) = \tilde{\mathfrak{p}}$. This implies $\mathrm{Im}(\pi^*) = V(\mathfrak{a})$.

. $\pi^*(\mathfrak{p}_1) = \pi^*(\mathfrak{p}_2) \Rightarrow \mathfrak{p}_1 = \dfrac{\pi^*(\mathfrak{p}_1)}{\mathfrak{a}} = \dfrac{\pi^*(\mathfrak{p}_2)}{\mathfrak{a}} = \mathfrak{p}_2$;

and so π^* is injective. ■

Remark. In fact π^* induces a homeomorphism from $\mathrm{Spec}(A/\mathfrak{a})$ to $V(\mathfrak{a})$ w.r.t. the induced Zariski topology on $V(\mathfrak{a})$. So far we have showed π^* gives us a continuous bijection. Notice $\pi^*(V(\mathfrak{b}/\mathfrak{a})) = V(\mathfrak{b})$, and so π^* is a closed map.

Lecture 02: Contraction map and localization

Wednesday, April 4, 2018 12:18 AM

<u>Proposition</u>. Suppose S is a multiplicatively closed subset of A, and $0 \notin S$. Let $f: A \longrightarrow S^{-1}A$, $f(a) := \frac{a}{1}$. Then f^* induces a bijection from $\operatorname{Spec}(S^{-1}A)$ to $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid S \cap \mathfrak{p} = \emptyset\}$.

(we will prove this in the next lecture.)