Homework 7

Thursday, May 31, 2018 10:23 PM

1. Let $\mathcal{O}_k$ be the ring of integers of a number field k. Prove that $\mathcal{O}_k$ is a UFD if and only if $\mathcal{O}_k$ is a PID.

2. Suppose k is an algebraically closed field. For $f_1, \ldots, f_{n-1}$ in $k[x_1, \ldots, x_{n-1}]$, let $X(f_1, \ldots, f_{n-1}) := \{ p \in k^n \mid f_1(p) = \cdots = f_{n-1}(p) = 0 \}$. Using Krull's height theorem and $\dim k[x_1, \ldots, x_n] = n$, prove that either $X(f_1, \ldots, f_{n-1}) = \emptyset$ or $|X(f_1, \ldots, f_{n-1})| = \infty$.

3. Suppose $\mathcal{O}$ is an integral domain and A is a finitely generated $\mathcal{O}$-algebra. Let $i : \mathcal{O} \hookrightarrow A$ and $i^* : \operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}$. For $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}$, let $k(\mathfrak{p}) :=$ the field of fractions of $\mathcal{O}/\mathfrak{p}$. Prove that $\exists \alpha \in \mathcal{O}$ s.t. $\alpha \notin \mathfrak{p} \Rightarrow \dim A \otimes_{\mathcal{O}} k(\mathfrak{p}) = \dim A \otimes_{\mathcal{O}} k(0)$.

[Hint: $\dim k[x_1, \ldots, x_n] = n$. Use Noether normalization for $A \otimes_{\mathcal{O}} k(0)$ to find $\alpha \in \mathcal{O} \setminus \{0\}$ and $x_1, \ldots, x_n \in A$ s.t. x_i's are alg. indep. over $k(0)$ and $A[1/\alpha]$ is integral over $\mathcal{O}[1/\alpha][x_1, \ldots, x_n]$. Deduce, if $\alpha \notin \mathfrak{p}$, then $A \otimes_{\mathcal{O}} k(\mathfrak{p})$ is integral over $k(\mathfrak{p})[x_1, \ldots, x_n]$.]

Homework 7

Saturday, June 2, 2018 10:48 PM

4. Suppose A is a Noetherian ring and $\mathfrak{a} \ntrianglelefteq A$. Let

$\mathfrak{b} := \bigcap_{i=1}^{\infty} \mathfrak{a}^{2^i}$. Prove that $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$.

[Hint. Suppose $\mathfrak{a}\mathfrak{b} \neq \mathfrak{b}$, and let $\bigcap_{j=1}^{n} \mathfrak{q}_j$ be a reduced primary decomposition of $\mathfrak{a}\mathfrak{b}$. So $\exists j$, $\mathfrak{b} \nsubseteq \mathfrak{q}_j$.

Suppose $x \in \mathfrak{b} \setminus \mathfrak{q}_j$. Then $\mathfrak{a} \subseteq (\mathfrak{q}_j : x) \subseteq \mathfrak{p}_j \Rightarrow$

$\mathfrak{b} \subseteq \mathfrak{a}^m \subseteq \mathfrak{p}_j^m \subseteq \mathfrak{q}_j$ which is a contradiction.]

5. Suppose A is a Noetherian local ring and $\operatorname{Max} A = \{\mathfrak{m}\}$.

Prove $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$.