Homework 5

Saturday, May 12, 2018 11:10 PM

1. Let k be a finite extension of $\mathbb{Q}$, and $\mathcal{O}_k$ be the integral closure of $\mathbb{Z}$ in k. In class we have proved that $\mathcal{O}_k \simeq \mathbb{Z}^d$ as an abelian group where $d = [k:\mathbb{Q}]$. Suppose $\mathcal{O}_k = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_d$; and $\mathrm{Hom}_{\mathbb{Q}}(k, \overline{\mathbb{Q}}) = \{\sigma_1, \ldots, \sigma_d\}$ where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. For $a \in k$, let $N_{k/\mathbb{Q}}(a) := \sigma_1(a)\,\sigma_2(a) \cdots \sigma_d(a)$.

(a) Prove that $D_k := \det[\sigma_i(a_j)]^2 \in \mathbb{Z}$. (It is called the discriminant of k.)

(b) Prove that for any $a \in \mathcal{O}_k$, $|N_{k/\mathbb{Q}}(a)| = [\mathcal{O}_k : a\mathcal{O}_k]$.

(This justifies $N_{k/\mathbb{Q}}(\mathfrak{a}) := |\mathcal{O}_k/\mathfrak{a}|$ for $\mathfrak{a} \lhd \mathcal{O}_k$.)

2. Suppose A is a valuation ring of a field F, and $A \subseteq A' \subseteq F$ is a subring. Suppose $\mathrm{Max}\, A = \{\mathfrak{m}\}$ and $\mathrm{Max}\, A' = \{\mathfrak{m}'\}$. Prove that (a) $\mathfrak{m}' \subseteq \mathfrak{m}$

(b) $\mathfrak{m}' \in \mathrm{Spec}(A)$ and $A' = A_{\mathfrak{m}'}$

(c) $A/\mathfrak{m}'$ is a valuation ring of $A'/\mathfrak{m}'$.

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3. (a) Suppose Γ is a totally ordered abelian group, and F is a field.

A valuation of F is a function $v: F \to \Gamma \cup \{\infty\}$ with the following properties:

- $v(a) = \infty \iff a = 0$
- $\forall \gamma \in \Gamma$, $\gamma < \infty$, $\gamma + \infty = \infty$, $\infty + \infty = \infty$.
- $v(xy) = v(x) + v(y) \quad \forall x, y \in F$.
- $v(x+y) \geq \min\{v(x), v(y)\} \quad \forall x, y \in F$.

Let $\mathcal{O}_v := \{x \in F \mid v(x) \geq 0\}$ and $\mathfrak{m}_v := \{x \in F \mid v(x) > 0\}$.

Prove that $\mathcal{O}_v$ is a valuation ring of the field F, and $\mathrm{Max}(\mathcal{O}_v) = \{\mathfrak{m}_v\}$.

(b) Let A be a valuation ring of a field F. Let $\Gamma := F^\times / A^\times$.

We say $xA^\times \geq yA^\times$ if $xy^{-1} \in A$. Prove that Γ is a totally ordered abelian group, $v: F \to \Gamma \cup \{\infty\}$,

$$v(x) = \begin{cases} xA^\times & x \in F^\times \\ \infty & x = F \end{cases}$$

is a valuation of F, and $\mathcal{O}_v = A$.

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4. (a) Suppose A is a local Noetherian ring with maximal ideal $\mathfrak{m}$, and M is a finitely generated A-module. Prove that

$$M \text{ is flat} \iff M \text{ is free}$$

(b) Suppose A is a Noetherian ring, and M is a f.g. A-mod. Prove that the following are equivalent:

(b-1) M is flat.

(b-2) $\forall \mathfrak{p} \in \operatorname{Spec} A$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-mod.

(b-3) $\forall \mathfrak{m} \in \operatorname{Max} A$, $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-mod.

(Hint. (a) ($\Longrightarrow$) Suppose $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is an $A/\mathfrak{m}$-basis of $M/\mathfrak{m}M$. Using Nakayama's lemma show $M = \langle x_1, \ldots, x_n \rangle$. Consider the S.E.S.

$$0 \to K \to A^n \to M \to 0$$

Use Math200b, HW 6, problem 4 and deduce

$$0 \to K \otimes_A A/\mathfrak{m} \to A^n \otimes_A A/\mathfrak{m} \to M \otimes_A A/\mathfrak{m} \to 0$$

is a S.E.S.. Conclude that $K \otimes_A A/\mathfrak{m} \simeq K/\mathfrak{m}K = 0$. Using Nakayama's lemma deduce $K = 0$.)