Homework 1

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$\boxed{1}$ Let $f: A \to B$ be a ring homomorphism. Let $\underline{\text{Ideal}}(A)$ and $\underline{\text{Ideal}}(B)$ be the set of ideals of A and B, resp.

Then as we discussed in class

$$\underline{\text{Ideal}}(B) \xrightarrow{c} \underline{\text{Ideal}}(A), \quad \mathfrak{b} \longmapsto \mathfrak{b}^c := f^{-1}(\mathfrak{b})$$
(contraction)

$$\underline{\text{Ideal}}(A) \xrightarrow{e} \underline{\text{Ideal}}(B), \quad \mathfrak{a} \longmapsto \mathfrak{a}^e := \langle f(\mathfrak{a}) \rangle$$
(extension)

are two functions. Prove that

$$\mathfrak{a}^{ece} = \mathfrak{a}^e \quad \text{and} \quad \mathfrak{b}^{cec} = \mathfrak{b}^c;$$

And deduce that the extension and the contraction maps induce bijections between $\mathrm{Im}(e)$ and $\mathrm{Im}(c)$.

$\boxed{2}$ Suppose $\mathfrak{a}, \mathfrak{b} \lhd A$. Let $(\mathfrak{a} : \mathfrak{b}) := \{x \in A \mid x\mathfrak{b} \subseteq \mathfrak{a}\}$.

(a) Prove that $(\mathfrak{a} : \mathfrak{b}) \lhd A$ and $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$

(b) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$, (c) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$

d) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$ e) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap (\mathfrak{a} : \mathfrak{b}_i)$.

$\boxed{3}$ Recall that $\sqrt{\mathfrak{a}} := \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{Z}^+\}$.

(a) Prove that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

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(b) Prove that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime if and only if $\sqrt{\mathfrak{a}}$ and $\sqrt{\mathfrak{b}}$ are coprime.

$\boxed{4}$ (a) Prove that $\mathrm{Nil}(A[x]) = \mathrm{Nil}(A)[x]$.

(b) Prove that $U(A[x]) = \left\{ a_0 + a_1 x + \cdots + a_n x^n \,\middle|\, \begin{array}{l} n \in \mathbb{Z}^{\geq 0} \\ a_0 \in U(A) \\ a_1, \ldots, a_n \in \mathrm{Nil}(A) \end{array} \right\}$.

(c) Prove that $J(A[x]) = \mathrm{Nil}(A)[x]$.

(Here $A[x]$ is the ring of polynomials over A with indeterminant x.)

$\boxed{5}$ (a) Prove that $\{\mathfrak{p}\}$ is closed in $\mathrm{Spec}(A) \iff \mathfrak{p} \in \mathrm{Max}(A)$.

(b) Prove that the closure $\overline{\{\mathfrak{p}\}}$ of $\{\mathfrak{p}\}$ in $\mathrm{Spec}(A)$ is $V(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$.

$\boxed{6}$ Let $X = \mathrm{Spec}(A)$ and, for $f \in A$, let $X_f := X \setminus V(\langle f \rangle)$.

(a) Prove that $X_f = X_{f'} \iff \sqrt{\langle f \rangle} = \sqrt{\langle f' \rangle}$.

(b) Prove that there is a bijection between X_f and $\mathrm{Spec}(A_f)$ where $A_f := S_f^{-1} A$ and $S_f = \{1, f, f^2, \ldots\}$. (We consider spec. of the $\underline{0}$ ring to be $\emptyset$.)

(c) Prove that $\{X_f\}_{f \in A}$ is a basis of open sets of X.

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(d) Prove that X is quasi-compact; that means every open covering of X has a finite sub-cover.

$\boxed{7}$ Suppose $\mathfrak{a}, \mathfrak{b}_1, \ldots, \mathfrak{b}_n \triangleleft A$, $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{b}_i$, and $\mathfrak{a} \not\subseteq \bigcup_{\substack{i=1 \\ i\neq j}}^{n} \mathfrak{b}_i$ for any $1 \leq j \leq n$. Prove that for some $k \in \mathbb{Z}^+$

$$\mathfrak{a}^k \subseteq \bigcap_{i=1}^{n} \mathfrak{b}_i .$$

(Hint. • $\mathfrak{a} \subseteq \mathfrak{b}_1 \cup \mathfrak{b}_2 \Rightarrow \mathfrak{a} \subseteq \mathfrak{b}_i$ for some i.

• $\mathfrak{a} \subseteq (\mathfrak{b}_1 + \mathfrak{b}_2) \cup \mathfrak{b}_3 \cup \cdots \cup \mathfrak{b}_n$; by induction deduce

$\mathfrak{a}^k \subseteq \prod_{i<j} (\mathfrak{b}_i + \mathfrak{b}_j)$ (I)

• Show that $\mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_{n-1} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_n$ (II)

• (I), (II) imply $\mathfrak{a}^k \subseteq \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_n$.)