

Lemma 1. Suppose $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ are A -module homomorphisms. Then the following statements hold:

1. If $h_M(N_1) \xrightarrow{h_M(f_1)} h_M(N_2) \xrightarrow{h_M(f_2)} h_M(N_3)$ is an exact sequence for every A -module M , then $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ is an exact sequence.
2. If $h^M(N_1) \xleftarrow{h^M(f_1)} h^M(N_2) \xleftarrow{h^M(f_2)} h^M(N_3)$ is an exact sequence for every A -module M , then $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ is an exact sequence.

Proof. In each part, we must prove that $\ker f_2 = \operatorname{Im} f_1$.

(1) Applying the assumption for $M = N_1$, we deduce that

$$0 = (h_{N_1}(f_2) \circ h_{N_1}(f_1))(1_{N_1}) = f_2 \circ f_1 \circ 1_{N_1} = f_2 \circ f_1,$$

which implies that $\operatorname{Im} f_1 \subseteq \ker f_2$.

Next, we apply the assumption for $M = \ker f_2$. Note that

$$(h_{\ker f_2}(f_2))(i) = f_2 \circ i = 0,$$

where $i: \ker f_2 \rightarrow N_2$ is the inclusion map. Hence,

$$i \in \ker((h_{\ker f_2}(f_2))) = \operatorname{Im}((h_{\ker f_2}(f_1))),$$

which means for some $g: \ker f_2 \rightarrow N_1$ such that

$$i = (h_{\ker f_2}(f_1))(g) = f_1 \circ g.$$

Hence, $\ker f_2 \subseteq \operatorname{Im} f_1$. This completes the proof of the first part.

(2) Applying the assumption for $M = N_3$, we deduce that

$$0 = (h^{N_3}(f_1) \circ h^{N_3}(f_2))(1_{N_3}) = 1_{N_3} \circ f_2 \circ f_1 = f_2 \circ f_1.$$

Therefore, $\operatorname{Im} f_1 \subseteq \ker f_2$.

Next, we apply the assumption for $M := N_2/\operatorname{Im} f_1$. Note that

$$(h^M(f_1))(\pi) = \pi \circ f_1 = 0,$$

where $\pi: N_2 \rightarrow N_2/\operatorname{Im} f_1$ is the natural quotient map. Thus,

$$\pi \in \ker(h^M(f_1)) = \operatorname{Im}(h^M(f_2)),$$

which means for some $g: N_3 \rightarrow N_2/\operatorname{Im} f_1$ such that

$$\pi = (h^M(f_2))(g) = g \circ f_2.$$

Therefore $\ker f_2 \subseteq \ker \pi = \operatorname{Im} f_1$. This completes the proof. $\square$