## 1 Homework 9.

1. Suppose  $M_i$ 's and N are A-modules,  $M_3$  is flat, and

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is a SES. Prove that

$$0 \to M_1 \otimes_A N \xrightarrow{f_1 \otimes \mathrm{id}_N} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}_N} M_3 \otimes_A N \to 0$$

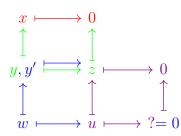
is a SES.

(Hint. Argue that there exists a SES

$$0 \to K \to F \to N \to 0$$

such that F is a free A-module. Discuss why the following is a commutating diagram where all the rows and columns are exact.

Suppose  $x \in M_1 \otimes_A N$  is in the kernel of  $f_1 \otimes id_N$ . Then use the following diagram:



Start with red, deduce existence of green, get the violet part, continue with blue. Argue why y = y'; and deduce that x = 0.)

2. Suppose  $0 \to M_1 \to M_2 \to M_3 \to 0$  is a SES of A-modules and  $M_3$  is flat. Prove that  $M_1$  is flat if and only if  $M_2$  is flat.

(Hint. Use the previous problem and the Short Five Lemma.)

- 3. Suppose E/F is a field extension,  $\alpha \in E$ , and  $[F[\alpha] : F]$  is odd. Prove that  $F[\alpha^2] = F[\alpha]$ .
- 4. Suppose  $a_1, \ldots, a_n$  are positive rational numbers. Prove that  $\sqrt[3]{2}$  is not in  $\mathbb{Q}[\sqrt{a_1}, \ldots, \sqrt{a_n}]$ .
- 5. Suppose  $E \subseteq \mathbb{C}$  is a splitting field of  $x^p 2$  over  $\mathbb{Q}$  where p is an odd prime number.
  - (a) Prove that  $E = \mathbb{Q}[\sqrt[p]{2}, \zeta_p]$  where  $\zeta_p := e^{2\pi i/p}$ .
  - (b) Prove that  $[E:\mathbb{Q}] = p(p-1)$ .

.(**Hint.** Notice that  $[E : \mathbb{Q}]$  is a multiple of  $[\mathbb{Q}[\zeta_p] : \mathbb{Q}]$  and  $[\mathbb{Q}[\sqrt[p]{2}] : \mathbb{Q}]$ . Argue that  $[\mathbb{Q}[\zeta_p] : \mathbb{Q}] = p - 1$  and  $[\mathbb{Q}[\sqrt[p]{2}] : \mathbb{Q}] = p$ .)

- 6. Suppose E is a splitting field of  $f(x) \in F[x]$  over F.
  - (a) Prove that if  $gcd(f, f') \neq 1$ , then  $E \otimes_F F[x]/\langle f \rangle$  has a non-zero nilpotent element.
  - (b) Prove that if gcd(f, f') = 1, then

$$E \otimes_F (F[x]/\langle f \rangle) \simeq \underbrace{E \oplus \cdots \oplus E}_{\text{deg }f\text{-times}}$$

7. Suppose p is an odd prime, and  $a \in \mathbb{F}_p^{\times}$ . Prove that  $x^p - x + a$  is irreducible in  $\mathbb{F}_p[x]$ .

(**Hint.** Let *E* be a splitting field of  $x^p - x + a$  over  $\mathbb{F}_p$ . Let  $\alpha \in E$  be a zero of  $x^p - x + a$ . Prove that  $\alpha + i$  is a zero of  $x^p - x + a$  for every  $i \in \mathbb{F}_p$ , and deduce that

$$x^p - x + a = \prod_{i \in \mathbb{F}_p} (x - \alpha - i).$$

Notice that  $m_{\alpha,\mathbb{F}_p}(x)$  divides  $x^p - x + a$ , and consider the coefficient of  $x^{d-1}$ in  $m_{\alpha,\mathbb{F}_p}(x)$  and show that deg  $m_{\alpha,\mathbb{F}_p} = p$ .)