

1 Homework 8.

1. Suppose B is an A -algebra. Let $F: B\text{-mod} \rightarrow A\text{-mod}$ be the forgetful functor.

- (a) Prove that

$$\text{Hom}_A(M, F(N)) \simeq \text{Hom}_B(T_B(M), N)$$

as A -modules.

- (b) Convince yourself that

$$(M, N) \mapsto \text{Hom}_A(M, F(N))$$

$$(M, N) \mapsto F(\text{Hom}_B(T_B(M), N))$$

can be extended to functors

$$G_1, G_2: (A\text{-mod})^{\text{op}} \times (B\text{-mod}) \rightarrow A\text{-mod}.$$

(You do not need to write anything for this part.)

- (c) Prove that $G_1 \simeq G_2$.

(**Remark.** We say the isomorphism in part (a) is natural on both M and N .)

2. Suppose A is a local unital commutative ring and $\mathfrak{a}$ is an ideal of A .

- (a) Suppose M is a flat A -module. Prove that $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$.

- (b) Suppose $0 \rightarrow M_1 \xrightarrow{i} M_2$ is injective, and M_1 and M_2 are flat A -modules. Prove that

$$\text{id}_{\mathfrak{a}} \otimes i : \mathfrak{a} \otimes_A M_1 \rightarrow \mathfrak{a} \otimes_A M_2$$

is injective.

- (c) Suppose $0 \rightarrow N_1 \xrightarrow{i} N_2 \rightarrow N_3 \rightarrow 0$ is a SES, and N_2 and N_3 are flat A -modules. Prove that

$$\mathfrak{a}N_2 \cap i(N_1) = i(\mathfrak{a}N_1).$$

- (d) Suppose $M := F/K$ where F is a free A -module and K is a submodule of F . Suppose M is a flat A -module. Prove that

$$\mathfrak{a}F \cap K = \mathfrak{a}K.$$

(**Hint.** For part (c), deduce that N_1 is a flat A -module.)

3. Suppose A is a local unital commutative ring and $\text{Max}(A) = \{\mathfrak{m}\}$.
- (a) Suppose K is a finitely generated submodule of A^n and $\mathfrak{m}^n \cap K = \mathfrak{m}K$. Prove that K is a free A -module and $A^n = K \oplus N$ for some finitely generated submodule N of A^n .
- (b) Suppose M is a finitely presented A -module; that means, for some positive integer n , there is a finitely generated submodule K of A^n such that $M \simeq A^n/K$. Suppose M is a flat A -module. Prove that M is free.

(**Hint.** (a) Notice that $0 \rightarrow \frac{K}{\mathfrak{m}K} \rightarrow \frac{A^n}{\mathfrak{m}^n}$ is injective of $(A/\mathfrak{m})$ -vector spaces. Hence there are $x_1, \dots, x_m \in K$ and $x_{m+1}, \dots, x_n \in A^n$ such that $\bar{x}_i := x_i + \mathfrak{m}K$, for $i = 1..m$ is a $(A/\mathfrak{m})$ -basis of $\frac{K}{\mathfrak{m}K}$ and $\bar{x}'_i := x_i + \mathfrak{m}^n$, for $i = 1..n$ is a $(A/\mathfrak{m})$ -basis of $A^n/\mathfrak{m}^n$. Use Nakayama's lemma and show that

$$K = \bigoplus_{i=1}^m Ax_i \quad \text{and} \quad A^n = \bigoplus_{i=1}^n Ax_i.$$

- (b) Use Problem 1(d) and deduce that $\mathfrak{m}^n \cap K = \mathfrak{m}K$. Use part (a) and complete the proof.)
4. Suppose A is a unital commutative ring and M is a finitely presented flat A -module. Prove that the following statements are equivalent:
- (a) M is flat.
- (b) $M_{\mathfrak{p}}$ is flat for every $\mathfrak{p} \in \text{Spec}(A)$.
- (c) $M_{\mathfrak{p}}$ is free for every $\mathfrak{p} \in \text{Spec}(A)$.
- (d) M is projective.

(**Remark.** Last week you have showed: (a) $\Leftrightarrow$ (b) (without f.p. assumption). Problem 2(b) shows that (b) $\Leftrightarrow$ (c) and (c) $\Rightarrow$ (b) holds without f.p.

assumption. You have already proved that (c) $\Leftrightarrow$ (d). Note that (d) $\Rightarrow$ (a) without f.p. assumption.)

5. Suppose M is a flat A -module. For $x_1, \dots, x_n \in M$, suppose

$$a_1x_1 + \dots + a_nx_n = 0.$$

Let $\mathfrak{a} := \langle a_1, \dots, a_n \rangle$ be the ideal generated by a_i 's. Let

$$f : A^n \rightarrow \mathfrak{a}, \quad f(b_1, \dots, b_n) := a_1b_1 + \dots + a_nb_n,$$

$K := \ker f$, and so

$$0 \rightarrow K \rightarrow A^n \rightarrow I \rightarrow 0 \tag{1}$$

is a short exact sequence of A -modules.

Prove that there exist $\mathbf{k}_j := (b_{1j}, \dots, b_{nj}) \in K$ and $y_j \in M$ for $j = 1, \dots, m$ such that

$$b_{i1}y_1 + \dots + b_{im}y_m = x_i$$

for every $i = 1, \dots, n$; that means

$$\mathbf{k}_1y_1 + \dots + \mathbf{k}_my_m = (x_1, \dots, x_n).$$

(**Hint.** Recall that $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$ with an isomorphism which sends $a \otimes x$ to ax for every $a \in \mathfrak{a}$ and $x \in M$. Use flatness and the SES in (1).)

(**Remark.** This is called the *equation criterion*)

6. Let $A_p := \mathbb{Z}[x]/\langle x^2 + x + 1 \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

- (a) Prove that A_p is a field if and only if $p \not\equiv 1 \pmod{3}$ and $p \neq 3$.
- (b) Prove that $A_p \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ as rings if and only if $p \equiv 1 \pmod{3}$.
- (c) Prove that A_p has a non-zero nilpotent element if and only if $p = 3$.

7. Prove that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic (as fields).

8. Suppose $\alpha, \beta \in \mathbb{C}$ are algebraic over $\mathbb{Q}$. Let $f := m_{\alpha, \mathbb{Q}}$ and $g := m_{\beta, \mathbb{Q}}$. Prove that f is irreducible in $(\mathbb{Q}[\beta])[x]$ if and only if g is irreducible in $(\mathbb{Q}[\alpha])[x]$.

(**Hint.** Prove that both of these conditions are equivalent to

$$[\mathbb{Q}[\alpha, \beta] : \mathbb{Q}] = \deg f \deg g.)$$