

1 Homework 7.

1. Prove that $\langle 2, \sqrt{-10} \rangle$ is a projective $\mathbb{Z}[\sqrt{-10}]$ -module.
2. Prove that if $h_x \simeq h_y$, then $x \simeq y$.
3. For given functors $F_1, F_2: \mathcal{C} \rightarrow \mathcal{C}$, define the following:

$$\begin{aligned} G_i: \mathcal{C} \times \mathcal{C}^{\text{op}} &\rightarrow \underline{\text{Set}}, & G_i(x, y) &:= h^x(F_i(y)), \\ G_i((f, g): (x, y) \rightarrow (x', y')) &: h^x(F_i(y)) \rightarrow h^{x'}(F_i(y')), \\ G_i((f, g))(h) &:= f \circ h \circ F_i(g). \end{aligned}$$

- (a) Prove that G_i is a functor.
- (b) Prove that $G_1 \simeq G_2$ if and only if $F_1 \simeq F_2$

(You do not need to write any argument for this problem.)

(**Remark.** We simply say if $h^x \circ F_1 \simeq h^x \circ F_2$ naturally, then $F_1 \simeq F_2$.)

4. Suppose A is a unital commutative ring, $S \subseteq A$ is a multiplicatively closed subset, and M is an A -module.
 - (a) Convince yourself that localizing defines an exact functor from $\underline{A\text{-mod}}$ to $\underline{S^{-1}A\text{-mod}}$. (You do not need to write any argument for this part.)
 - (b) Prove that $S^{-1}A \otimes_A M \simeq S^{-1}M$; deduce that $S^{-1}A$ is a flat A -module.
 - (c) Prove that, if M is a flat A -module, then $S^{-1}M$ is a flat $S^{-1}A$ -module.
 - (d) Prove that $\frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}$ gives us a well-defined $S^{-1}A$ -module isomorphism

$$S^{-1}(M_1 \otimes_A M_2) \xrightarrow{\sim} S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

- (e) Prove that, if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ -module for every $\mathfrak{p} \in \text{Spec}(A)$, then M is flat. (Hint: look at previous HWs on localizing a module.)
5. Suppose A is a unital commutative ring and K is a field.

- (a) Suppose V and W are two K -vector spaces. Prove that

$$\dim_K(V \otimes_K W) = (\dim_K V)(\dim_K W)$$

(Hint. Use last week's problem 8.)

- (b) Suppose A is a local ring. Suppose M and N are finitely generated A -modules, and $M \otimes_A N = 0$. Prove that either $M = 0$ or $N = 0$.

(Hint. Suppose $\text{Max}(A) = \{\mathfrak{m}\}$. Let $k := A/\mathfrak{m}$. Argue

$$M/\mathfrak{m}M \simeq M \otimes_A k \quad \text{and} \quad N/\mathfrak{m}N \simeq N \otimes_A k.$$

Show that $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$. Use Nakayama's lemma.)

- (c) Recall that $\text{supp}(M) := \{\mathfrak{p} \in \text{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$. Suppose M and N are finitely generated A -modules. Prove that

$$\text{supp}(M \otimes_A N) = \text{supp}(M) \cap \text{supp}(N)$$

(**Remark.** Note that $\text{supp}(\mathbb{Z}/n\mathbb{Z}) = \{p\mathbb{Z} \mid p \text{ prime factor of } n\}$. Using the third part, you obtain an alternative way of showing $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = 0$ if and only if $\gcd(m, n) = 1$.)

6. Suppose A is a unital commutative ring, and M is an A -module. Let

$$T_M : \underline{A\text{-mod}} \rightarrow \underline{A\text{-mod}}, \quad T_M(N) := M \otimes_A N \quad \text{and} \quad T_M(f) := \text{id}_M \otimes f,$$

for $f \in \text{Hom}_A(N, N')$.

- (a) Prove that T_M is a functor.
 (b) Prove that there exists a natural isomorphism between $T_{M_1} \circ T_{M_2}$ and $T_{M_1 \otimes_A M_2}$.

(**Hint.** For $x_1 \in M_1$, let

$$f_{x_1} : M_2 \times N \rightarrow (M_1 \otimes_A M_2) \otimes N, \quad f_{x_1}(x_2, y) := (x_1 \otimes x_2) \otimes y.$$

Prove that f_{x_1} is A -bilinear. Deduce that there exists an A -module homomorphism

$$\phi_{x_1} : M_2 \otimes N \rightarrow (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi_{x_1}(x_2 \otimes y) = (x_1 \otimes x_2) \otimes y.$$

Let

$$f : M_1 \times (M_2 \otimes N) \rightarrow (M_1 \otimes_A M_2) \otimes N, \quad f(x_1, z) := \phi_{x_1}(z).$$

Prove that f is A -bilinear. Deduce that there exists an A -module homomorphism

$$\phi : M_1 \otimes_A (M_2 \otimes_A N) \rightarrow (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi(x_1 \otimes z) = f(x_1, z);$$

in particular, for every $x_1 \in M_1$, $x_2 \in M_2$, and $y \in N$,

$$\phi(x_1 \otimes (x_2 \otimes y)) = (x_1 \otimes x_2) \otimes y.$$

Similarly there exists an A -module homomorphism

$$\psi : (M_1 \otimes_A M_2) \otimes N \rightarrow M_1 \otimes_A (M_2 \otimes_A N),$$

such that

$$\psi((x_1 \otimes x_2) \otimes y) = x_1 \otimes (x_2 \otimes y).$$

Deduce that ϕ and ψ are inverse of each other. Convince yourself that this is a natural isomorphism. You do not need to include that in your solution.)

(Remark. We say that $M_1 \otimes_A (M_2 \otimes_A N)$ and $(M_1 \otimes_A M_2) \otimes_A N$ are naturally isomorphic.)

7. Suppose M_i 's and N are A -modules, M_3 is flat, and

$$0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$$

is a SES. Prove that

$$0 \rightarrow M_1 \otimes_A N \xrightarrow{f_1 \otimes \text{id}_N} M_2 \otimes_A N \xrightarrow{f_2 \otimes \text{id}_N} M_3 \otimes_A N \rightarrow 0$$

is a SES.

(Hint. Argue that there exists a SES

$$0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$$

such that F is a free A -module. Discuss why the following is a commutating

diagram where all the rows and columns are exact.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & M_1 \otimes_A N & \longrightarrow & M_2 \otimes_A N & \longrightarrow & M_3 \otimes_A N \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_1 \otimes_A F & \longrightarrow & M_2 \otimes_A F & \longrightarrow & M_3 \otimes_A F \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K \longrightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & 0
 \end{array}$$

Suppose $x \in M_1 \otimes_A N$ is in the kernel of $f_1 \otimes \text{id}_N$. Then use the following diagram:

$$\begin{array}{ccccc}
 x & \xrightarrow{\text{red}} & 0 & & \\
 \uparrow \text{green} & & \uparrow \text{green} & & \\
 y, y' & \xrightarrow{\text{green}} & z & \xrightarrow{\text{violet}} & 0 \\
 \uparrow & & \uparrow & & \uparrow \\
 w & \xrightarrow{\text{blue}} & u & \xrightarrow{\text{violet}} & ? = 0
 \end{array}$$

Start with **red**, deduce existence of **green**, get the **violet** part, continue with **blue**. Argue why $y = y'$; and deduce that $x = 0$.)

8. Suppose $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a SES of A -modules and M_3 is flat. Prove that M_1 is flat if and only if M_2 is flat.

(**Hint.** Use the previous problem and the Short Five Lemma.)