

1 Homework 6.

1. Suppose D is a local Noetherian integral domain.
 - (a) Prove that every submodule of a finitely generated projective D -module is projective if and only if D is a PID.
 - (b) Find a local Noetherian integral domain which is not a PID.
 - (c) Show that a submodule of a finitely generated projective module is not necessarily projective.

(**Hint.** Use two results from last week (1) submodule of a finitely generated free D -module is free if and only if D is a PID, (2) finitely generated projective modules of a local Noetherian ring are free.)

2. Suppose A is a unital commutative ring and $\text{Spec } A$ is the set of all the prime ideals of A . For an A -module M and $\mathfrak{p} \in \text{Spec } A$, let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$. Let

$$\text{supp } M := \{\mathfrak{p} \in \text{Spec } A \mid M_{\mathfrak{p}} \neq 0\}.$$

- (a) Prove that for a finitely generated A -module M ,

$$\text{supp } M = \{\mathfrak{p} \in \text{Spec } A \mid \text{Ann}(M) \subseteq \mathfrak{p}\},$$

where $\text{Ann}(M)$ is the annihilator of M .

- (b) For a positive integer d , let

$$U_d(M) := \{\mathfrak{p} \in \text{Spec } A \mid M_{\mathfrak{p}} \text{ can be generated by } d \text{ elements}\}.$$

Prove that if M is finitely generated and $\mathfrak{p}_0 \in U_d(M)$, then there exists $a_0 \in A \setminus \mathfrak{p}_0$ such that

$$\{\mathfrak{q} \in \text{Spec } A \mid a_0 \notin \mathfrak{q}\} \subseteq U_d(M).$$

(**Hint.** Suppose $N_{\mathfrak{p}_0} = M_{\mathfrak{p}_0}$ where $N = \langle x_1, \dots, x_d \rangle \subseteq M$. Consider $\text{supp}(M/N)$.)

- (c) Deduce that there exists an ideal $\mathfrak{a}_d(M)$ of A such that

$$U_d(M) = \text{Spec}(A) \setminus V(\mathfrak{a}_d(M)).$$

(Remark. This means $U_d(M)$ is Zariski-open)

3. We say an A -module M is finitely presented if there is a standard SES of the form

$$0 \rightarrow K \rightarrow A^n \xrightarrow{f} M \rightarrow 0, \text{ where } f(c_1, \dots, c_n) = \sum_{i=1}^n c_i x_i \text{ and } K = \ker f \quad (1)$$

such that K is a finitely generated A -module. Prove that if M is a finitely presented projective A -module, then for every $\mathfrak{p} \in \operatorname{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module and $\operatorname{rank}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq n$.

(Remark. Finitely presented projective modules are locally free.)

4. Let M be a finitely presented A -module given by the standard SES (1). Suppose $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $U_d(\cdot)$ be as in Problem 2. Let

$$U_d^{\circ}(M) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid d(M_{\mathfrak{p}}) = d\}.$$

Prove that

$$U_d^{\circ}(M) = U_d(M) \cap U_{n-d}(K).$$

Deduce that

$$\operatorname{Spec}(A) = \bigsqcup_{d=0}^n U_d^{\circ}(M)$$

is a partition of $\operatorname{Spec}(A)$ into Zariski-open sets.

(Remark. This result shows that for a f.p. module M , $\mathfrak{p} \mapsto d(M_{\mathfrak{p}})$ is locally constant in the Zariski-topology as $U_d^{\circ}(M)$ is Zariski-open.)

5. Suppose M is a finitely presented A -module. Note that this means there exists an exact sequence of the form

$$A^m \rightarrow A^n \rightarrow M \rightarrow 0.$$

Suppose S is a multiplicatively closed subset of A . Prove that

$$S^{-1}\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N), \quad \text{where } \frac{\phi}{1} \mapsto S^{-1}\phi.$$

(**Hint** Justify yourself that $M \mapsto S^{-1}M$ and $f \in \text{Hom}_A(M, N) \mapsto S^{-1}f \in \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ is an exact functor from the category of A -modules to the category of $S^{-1}A$ -modules. Hence

$$S^{-1}A^m \rightarrow S^{-1}A^n \rightarrow S^{-1}M \rightarrow 0$$

is an exact sequence. Since the contravariant representable functors are left-exact, we obtain the following exact sequences:

$$\begin{aligned} 0 \rightarrow h^N(M) \rightarrow h^N(A^n) \rightarrow h^N(A^m) \\ 0 \rightarrow h^{S^{-1}N}(S^{-1}M) \rightarrow h^{S^{-1}N}(S^{-1}A^n) \rightarrow h^{S^{-1}N}(S^{-1}A^m). \end{aligned}$$

Note that $h^N(A^k) \simeq N^k$, and deduce that the following are exact

$$\begin{aligned} 0 \rightarrow S^{-1}h^N(M) \rightarrow S^{-1}A^n \rightarrow S^{-1}A^m, \\ 0 \rightarrow h^{S^{-1}N}(S^{-1}M) \rightarrow S^{-1}A^n \rightarrow S^{-1}A^m. \end{aligned}$$

This implies that $S^{-1}h^N(M) \simeq h^{S^{-1}N}(S^{-1}M)$.)

6. Suppose M is a f.p. locally free A -module. Prove that M is projective.

(**Hint.** Suppose $f: N \rightarrow N'$ is a surjective A -module. You must show that $h_M(f)$ is surjective. Recall that this is equivalent to showing surjectivity of $(h_M(f))_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(A)$. Use the previous problem and the following

$$\begin{array}{ccc} h_M(N)_{\mathfrak{p}} & \xrightarrow{h_M(f)_{\mathfrak{p}}} & h_M(N')_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ h_{M_{\mathfrak{p}}}(N_{\mathfrak{p}}) & \xrightarrow{h_{M_{\mathfrak{p}}}(f_{\mathfrak{p}})} & h_{M_{\mathfrak{p}}}(N'_{\mathfrak{p}}) \end{array}$$

diagram and deduce that $(h_M(f))_{\mathfrak{p}}$ is surjective

precisely when $h_{M_{\mathfrak{p}}}(f_{\mathfrak{p}})$ is surjective. Since $M_{\mathfrak{p}}$ is free, deduce that $h_{M_{\mathfrak{p}}}(f_{\mathfrak{p}})$ is surjective.)

(**Remark.** Based on problems 3 and 6, you see that a f.p. module is projective if and only if it is locally free.)

7. (Yoneda's lemma) Suppose $\underline{C}$ is a locally small category, $a \in \text{Obj}(\underline{C})$,

$$h_a : \underline{C} \rightarrow \underline{\text{Set}}$$

is a representable functor, and $F : \underline{C} \rightarrow \underline{\text{Set}}$ is a functor. Let $\text{Nat}(h_a, F)$ be the class of all natural transformations from h_a to F .

- (a) Prove that the following is a bijection:

$$\theta_a : \text{Nat}(h_a, F) \rightarrow F(a), \quad \theta(\eta) := \eta_a(1_a).$$

- (b) For $f \in \text{Hom}_{\underline{C}}(a, a')$, let $\widehat{f} : h'_a \rightarrow h_a$ be

$$\widehat{f}_b(a' \xrightarrow{g} b) := a \xrightarrow{g \circ f} b.$$

Prove that $\widehat{f}$ is a natural transformation.

- (c) Prove that θ_a is a natural bijection; that means, if $f \in \text{Hom}_{\underline{C}}(a, a')$, then the following is a commuting diagram

$$\begin{array}{ccc} \text{Nat}(h_a, F) & \xrightarrow{\theta_a} & F(a) \\ \psi(f) \downarrow & & \downarrow F(f) \\ \text{Nat}(h_{a'}, F) & \xrightarrow{\theta_{a'}} & F(a') \end{array}$$

where for every $b \in \text{Obj}(\underline{C})$,

$$\psi(f)(\eta) := \eta \circ \widehat{f}.$$

8. (You do not have to write anything for this problem; only justify and understand all the statements. This is a useful result to have in your toolbox.)

For two functors F_1 and F_2 , we say $F_1 \simeq F_2$ if there exists a natural isomorphism $\eta : F_1 \rightarrow F_2$. Suppose $\{M_i\}_{i \in I}$ is a family of A -modules and N is an A -module.

- (a) Suppose $\{F_i\}_{i \in I}$ is a family of functors from $\underline{A\text{-mod}}$ to itself. Define the functor $\prod_{i \in I} F_i$.
- (b) Prove that

$$\prod_{i \in I} h_{M_i} \simeq h_{\bigoplus_{i \in I} M_i}.$$

- (c) Justify why we have

$$\begin{aligned} h_{\bigoplus_{i \in I} N \otimes_A M_i} &\simeq \prod_{i \in I} h_{N \otimes_A M_i} \simeq \prod_{i \in I} (h_{M_i} \circ h_N) \\ &\simeq \left(\prod_{i \in I} h_{M_i} \right) \circ h_N \simeq h_{\bigoplus_{i \in I} M_i} \circ h_N \\ &\simeq h_{N \otimes_A (\bigoplus_{i \in I} M_i)}. \end{aligned}$$

- (d) Prove that $\bigoplus_{i \in I} (N \otimes_A M_i) \simeq N \otimes_A (\bigoplus_{i \in I} M_i)$ as A -modules.