# 1 Homework 3.

1. Suppose $D$ is a PID and $M$ is a finitely generated $D$-module. Suppose

$$M \simeq D^r \oplus \bigoplus_{i=1}^{m} D/\langle a_i \rangle,$$

and $a_1, \ldots, a_m$ are the invariant factors of $M$. Prove that $d(M) = r + m$.

(**Remark.** In class, we proved that $\operatorname{rank}(M) = r$.)

2. Suppose $A$ is a unital commutative ring. A matrix $a \in \mathrm{M}_n(A)$ is called nilpotent if $a^m = 0$ for some non-negative integer $m$. Suppose $F$ is a field and $a \in \mathrm{M}_n(F)$ is a nilpotent matrix.

   (a) Prove that the annihilator of $V_a$ is of the form $\langle x^m \rangle$ for some $m \in \mathbb{Z}^+$.

   (b) Prove that $V_a \simeq F[x]/\langle x^{m_1} \rangle \oplus \cdots \oplus F[x]/\langle x^{m_k} \rangle$ for some positive integers $m_1 \leq \cdots \leq m_k$.

   (c) In the above setting, prove that $n = m_1 + \cdots + m_k$, and deduce that $a^n = 0$.

   (d) Prove that $\dim_F \ker(a^\ell) = \sum_{i=1}^k \min\{\ell, m_i\}$.

   (e) Prove that two nilpotent matrices $a_1, a_2 \in \mathrm{M}_n(a)$ are similar if and only if $\dim_F \ker(a_1^\ell) = \dim_F \ker(a_2^\ell)$ for every positive integer $\ell$.

3. Suppose $A$ is a unital commutative ring and $\mathrm{Nil}(A) = \{0\}$. Prove that $a \in \mathrm{M}_n(A)$ is nilpotent if and only if $a^n = 0$.

   (**Hint**. Use the previous problem, part (c), to show the claim for fields. Using field of fractions, obtain a similar result for integral domains. Then for every $\mathfrak{p} \in \mathrm{Spec}(A)$, consider $a$ modulo $\mathfrak{p}$ in $\mathrm{M}_n(A/\mathfrak{p})$.)

4. Let $D$ be a PID, and $F$ be a field of fractions of $D$. Suppose $a \in \mathrm{M}_{n,m}(D)$, and $r$ is the rank of $a$ as an element of $\mathrm{M}_{n,m}(F)$. Let

$$\underline{\ker a}(F) := \{v \in F^m \mid av = 0\}, \quad \underline{\ker a}(D) := \underline{\ker a}(F) \cap D^m, \quad \text{and,}$$

$$\underline{\mathrm{Im}\ a}(F) := \{av \in F^n \mid v \in F^m\}, \quad \underline{\mathrm{Im}\ a}(D) := \{av \in D^n \mid v \in D^m\}.$$

(a) Prove that $D^m/\underline{\ker a}(D)$ is a free $D$-module; and deduce that there exist $x_1, \ldots, x_m \in D^m$ such that

$$D^m = Dx_1 \oplus \cdots \oplus Dx_m, \quad \text{and}$$

$$\underline{\ker a}(D) = Dx_{r+1} \oplus \cdots \oplus Dx_m.$$

(b) Prove that there exist $y_1, \ldots, y_n \in D^n$ and $d_1, \ldots, d_r \in D \setminus \{0\}$ such that

$$D^n = Dy_1 \oplus \cdots \oplus Dy_n,$$

$$d_1 | \cdots | d_r, \quad \text{and}$$

$$\underline{\operatorname{Im} a}(D) = Dd_1y_1 \oplus \cdots \oplus Dd_ry_r.$$

(c) Let $x_i$'s be as in part (a). Prove that there exist $x'_1, \ldots, x'_r \in \bigoplus_{i=1}^r Dx_i$ such that

$$\bigoplus_{i=1}^r Dx_i = Dx'_1 \oplus \cdots \oplus Dx'_r, \quad \text{and}$$

$$ax'_i = d_iy_i$$

for all $i$.

(d) Prove that

$$\gamma_1 := [x'_1 \cdots x'_r x_{r+1} \cdots x_m] \in \mathrm{GL}_m(D),$$

$$\gamma_2 := [y_1 \cdots y_n] \in \mathrm{GL}_n(D), \quad \text{and}$$

$$a\gamma_1 = \gamma_2 \begin{pmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$a = \gamma_2 \begin{pmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix} \gamma_1^{-1}.$$

(**Remark.** This is called a Smith form of $A$.)

5. Let $a \in \mathrm{M}_n(\mathbb{Z})$, and $M_a := \mathbb{Z}^n/\underline{\operatorname{Im} a}(\mathbb{Z})$.

(a) Prove that $M_a$ is finite if and only if $\det a \neq 0$.

(b) Suppose $\det a \neq 0$. Prove that $|M_a| = |\det a|$.

(**Hint.** Suppose $a = \lambda_1 \begin{pmatrix} \mathrm{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix} \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathrm{GL}_n(\mathbb{Z})$ (a Smith form of $a$). Prove that

$$M_a \simeq \mathbb{Z}^{n-r} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}/d_i\mathbb{Z}.)$$

6. Suppose $F$ is a field, $a \in \mathrm{M}_n(F[x])$, and $\det a \neq 0$. Prove that

$$\dim_F(F[x]^n / \underline{\mathrm{Im}\ a}(F[x])) = \deg(\det a)).$$

(**Hint.** Suppose for some $\lambda_1, \lambda_2 \in \mathrm{GL}_n(F[x])$

$$a = \lambda_1 \begin{pmatrix} \mathrm{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix} \lambda_2$$

(a Smith form of $a$). Show that $n = r$, and

$$F[x]^n / \underline{\mathrm{Im}\ a}(F[x]) \simeq \bigoplus_{i=1}^{r} F[x]/\langle d_i(x) \rangle.)$$

7. Let $F$ be a field and $a \in \mathrm{M}_n(F)$. Suppose

$$xI - a = \gamma_1 \,\mathrm{diag}(f_1(x), \ldots, f_n(x))\gamma_2$$

is a Smith form of $xI - a \in \mathrm{M}_n(F[x])$; that means $\gamma_1, \gamma_2 \in \mathrm{GL}_n(F[x])$ and $f_1(x)|\cdots|f_n(x)$. Suppose $m$ is the largest integer such that $\deg f_{m-1} = 0$. Prove that $\mathrm{diag}(c(f_m), \ldots, c(f_n))$ is the rational canonical form of $a$.

(**Hint.** By the hint of the previous problem

$$F[x]/\underline{\mathrm{Im}\ (xI - a)}(F[x]) \simeq \bigoplus_{i=1}^{n} F[x]/\langle f_i(x) \rangle \simeq V_{\mathrm{diag}(c(f_m), \ldots, c(f_n))},$$

as $F[x]$-modules. Deduce that it is enough to prove

$$F[x]/\underline{\mathrm{Im}\ (xI - a)}(F[x]) \simeq V_a \tag{1}$$

as $F[x]$-modules. Let

$$\phi : F[x]^n \to V_a, \quad \phi(\sum_{i=0}^{m} x^i v_i) := \sum_{i=0}^{m} a^i v_i.$$

Argue why $\phi$ is an $F[x]$-module homomorphism. It is easy to see that $\underline{\mathrm{Im}\ (xI - a)}(F[x]) \subseteq \ker \phi$. Prove that equality holds, and deduce that (1) holds.)