Lecture 24: Algebraically closed extension

Thursday, March 1, 2018 7:53 PM

We were proving the following theorem:

<u>Theorem.</u> Let F be a field. Then there is a field extension E/F such that E is algebraically closed.

<u>Pf.</u> Let $A := F[x_f]_{\substack{f:\text{monic} \\ f\in F[x]\setminus F}}$ and $I = \langle f(x_f) \mid f:\text{monic}, f\in F[x]\setminus F\rangle$.

<u>Claim.</u> I is a proper ideal.

<u>Pf.</u> If not, $1\in I$. So $\exists\, g_1,\dots,g_m\in A$ and $f_1,\dots,f_m$ s.t.

$$g_1(x_f)\, f_1(x_{f_1}) + \cdots + g_m(x_f)\, f_m(x_{f_m}) = 1.$$

To make symbols more clear, let $y_i = x_{f_i}$ and $y_{m+1},\dots,y_n$ be the rest of variables involved in g_i's. So

$$(*)\quad g_1(y_1,\dots,y_n)\, f_1(y_1) + \cdots + g_m(y_1,\dots,y_n)\, f_m(y_m) = 1.$$

Let E' be the splitting field of $f_1(t)\cdot f_2(t)\cdots f_m(t)$ over F.

And let $\alpha_1,\dots,\alpha_m\in E'$ s.t. $f_1(\alpha_1) = f_2(\alpha_2) = \cdots = f_m(\alpha_m) = 0$.

Let's evaluate $(*)$ at $(\alpha_1,\dots,\alpha_m,0,\dots,0)$; and we get $0=1$, which is a contradiction.

Let $\mathfrak{m}$ be a maximal ideal of A s.t. $\mathfrak{m}\supseteq I$. Let $E_1 := A/\mathfrak{m}$.

So E_1 is a field; and $F\cap \mathfrak{m} = 0$ implies $F \hookrightarrow E_1$.

Lecture 24: Tower of algebraic extensions

Thursday, March 1, 2018 8:36 PM

<u>Claim</u>. Any monic polynomial $f(x) \in F[x] \setminus F$ has a zero in E_1.

<u>Pf</u>. Let $\alpha_f := x_f + \mathfrak{m} \in E$. Then $f(\alpha_f) = f(x_f) + \mathfrak{m} = \mathfrak{m}$

$\boxed{f(x_f) \in I \subseteq \mathfrak{m}}$.

We do the same construction again and again to get a tower of field extensions: $F \subseteq E_1 \subseteq E_2 \subseteq \cdots$

Let $E := \bigcup_{i=1}^{\infty} E_i$.

<u>Claim</u>. E is a field.

<u>Pf</u>. $\alpha \in E$, $\beta \in E \setminus \{0\} \Rightarrow \exists\, i$ s.t. $\alpha \in E_i$ and $\beta \in E_i \setminus \{0\}$.

So $\alpha \pm \beta, \alpha\beta^{\pm 1} \in E_i \Rightarrow \alpha \pm \beta, \alpha\beta^{\pm 1} \in E$.

<u>Claim</u>. E is algebraically closed.

<u>Pf</u>. Let $f(x) := \sum a_i x^i \in E[x]$. Then $\exists\, j$ s.t. $f(x) \in E_j[x]$.

$\Rightarrow f(x)$ has a zero in $E_{j+1} \Rightarrow f(x)$ has a zero in E. ■

<u>Proposition</u>. Suppose E/F and K/E are algebraic extensions. Then K/F is an algebraic extension.

<u>Pf</u>. Let $\alpha \in K$. Then $\alpha^n + e_{n-1}\alpha^{n-1} + \cdots + e_0 = 0$ for some e_i's $\in E$.

Since e_i's are algebraic over F, $[F[e_0, \ldots, e_{n-1}] : F] < \infty$.

Since α is algebraic over $F[e_0, \ldots, e_{n-1}]$,

$[F[e_0, \ldots, e_{n-1}, \alpha] : F[e_0, \ldots, e_{n-1}]] < \infty$. Hence $F[e_0, \ldots, e_{n-1}, \alpha]/F$ is a finite extension. Therefore α is algebraic over F. ■

Lecture 24: Algebraic closure

Thursday, March 1, 2018 8:51 PM

Theorem. Let F be a field. Then there is an algebraic field extension E/F such that E is algebraically closed.

Pf. Let $\tilde{E}/F$ be a field extension such that $\tilde{E}$ is algebraically closed (there is such a field by the previous theorem). Let E be the algebraic closure of F in $\tilde{E}$. So E/F is an algebraic field extension.

Claim E is algebraically closed.

Pf. Let $f(x) \in E[x]$. Since $\tilde{E}$ is algebraically closed, $\exists \alpha \in \tilde{E}$ s.t. $f(\alpha)=0$. So $E[\alpha]/E$ is algebraic. Since E/F is algebraic, by the previous proposition we deduce that $E[\alpha]/F$ is algebraic. Hence α is algebraic over F; this implies $\alpha \in E$; and claim follows. $\blacksquare$

Def. We call E an algebraic closure of F if E/F is algebraic and E is algebraically closed.

So far we have proved the existence of an algebraic closure. Next we show it is unique up to isomorphism.

Lecture 24: Uniqueness of algebraic closure

Thursday, March 1, 2018 9:32 PM

<u>Theorem</u>. (1) Let F be a field and Ω be an algebraically closed field. Suppose E is the splitting field of (monic) $f(x) \in F[x] \setminus F$ over F; and $\sigma: F \hookrightarrow \Omega$. Then $\exists \tilde{\sigma}: E \hookrightarrow \Omega$ s.t. $\tilde{\sigma}|_F = \sigma$.

(2) Let E and E' be two algebraic closures of F, and $\sigma_0: F \hookrightarrow E'$. Then $\exists\, \phi: E \xrightarrow{\simeq} E'$ st $\phi|_F = \sigma_0$.

<u>Pf.</u> (1) Since Ω is algebraically closed, $\sigma(f)(x) = (x-\omega_1)\cdots(x-\omega_n)$ for some $\omega_1, \ldots, \omega_n$. Then $E' := \sigma(F)[\omega_1, \ldots, \omega_n]$ is the splitting field of $\sigma(f)$ over $\sigma(F)$. Hence $\exists\, \tilde{\sigma}: E \xrightarrow{\simeq} E' \subseteq \Omega$ s.t. $\tilde{\sigma}|_F = \sigma$.

(2) Let $\Sigma := \{(K, \sigma) \mid \underset{\text{subfield}}{F \subseteq K \subseteq E},\ \underset{\sigma|_F = \sigma_0}{\sigma: K \hookrightarrow E'}\}$.

We say $(K_1, \sigma_1) \preccurlyeq (K_2, \sigma_2)$ if and only if $K_1 \subseteq K_2$ and $\sigma_2|_{K_1} = \sigma_1$.

$(\Sigma, \preccurlyeq)$ is POSet.

<u>Claim</u>. Σ has a maximal element.

<u>Pf.</u> By Zorn's lemma, it is enough to show any chain has an upper bound. Suppose $\{(K_i, \sigma_i)\}_{i \in I}$ is a chain. Let $K := \bigcup_{i \in I} K_i$ and

Lecture 24: Uniqueness of algebraic closure

Thursday, March 1, 2018 9:49 PM

$\sigma: K \to E'$, $\sigma(a) = \sigma_i(a)$ if $a \in K_i$. Notice that σ is well-def.

if $a \in K_i$ and $a \in K_j$, as $\{(K_\ell, \sigma_\ell)\}$ is a chain, w.l.o.g.

we can and will assume $(K_i, \sigma_i) \preccurlyeq (K_j, \sigma_j)$. Hence $K_i \subseteq K_j$

and $\sigma_j|_{K_i} = \sigma_i$. And so $\sigma_j(a) = \sigma_i(a)$.

- Show that $\bigcup_{i \in I} K_i$ is a field.

- $\forall\, a \in F \subseteq K_i$ $(\forall i \in I)$, $\sigma(a) = \sigma_i(a) = a$.

Hence $(K, \sigma) \in \Sigma$ and $(K_i, \sigma_i) \preccurlyeq (K, \sigma)$ for any $i \in I$.

. Therefore $(\Sigma, \preccurlyeq)$ has a maximal element. Suppose (K, σ) is a maximal element of Σ.

<u>Claim</u>. $K = E$.

(We will prove this in the next lecture.)