Lecture 20: Tensor product and direct sum

Tuesday, February 20, 2018 10:53 AM

<u>Proposition</u>. Consider the split short exact sequence

$$0 \longrightarrow N_1 \xrightarrow{j_1} N_1 \oplus N_2 \xrightarrow{p} N_2 \longrightarrow 0 . \quad \text{Then}$$

$$\begin{tikzcd}
0 \arrow[r] & N_1 \otimes_S M \arrow[r, "j \otimes id"] \arrow[d, equal] & (N_1 \oplus N_2) \otimes_S M \arrow[r, "p \otimes id"] & N_2 \otimes_S M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_1 \otimes_S M \arrow[r, "\bar{j}"'] \arrow[ur, phantom, "\circlearrowleft"] & (N_1 \otimes_S M) \oplus (N_2 \otimes_S M) \arrow[r, "\bar{p}"'] \arrow[u, "\phi", "\wr"'] & N_2 \otimes_S M \arrow[r] \arrow[ul, phantom, "\circlearrowleft"] & 0
\end{tikzcd}$$

where $\phi(n_1 \otimes m_1 \, , \, n_2 \otimes m_2) := j_1(n_1) \otimes m_1 + j_2(n_2) \otimes m_2$.

<u>Pf.</u> By the universal property of (external) direct sum $\exists$ an R-mod. hom. $\phi : (N_1 \otimes_S M) \oplus (N_2 \otimes_S M) \longrightarrow (N_1 \oplus N_2) \otimes_S M$,

$$\phi(n_1 \otimes m, 0) = j_1(n_1) \otimes m \, , \quad \phi(0, n_2 \otimes m) = j_2(n_2) \otimes m .$$

Now one can see that the above diagram commutes. Hence by the Short Five Lemma ϕ is an isomorphism. ∎

Here is an alternative approach: let $\prod_{i \in I} F_{N_i} : S\text{-mod} \longrightarrow Ab$

Then in your HW assignments you have essentially proved that there is a natural transformation from $F_{\bigoplus_{i \in I} N_i}$ to $\prod_{i \in I} F_{N_i}$.

$$\Big(\prod_{i \in I} F_{N_i}\Big)(K) := \prod_{i \in I} F_{N_i}(K)$$

$$\begin{tikzcd}
K \arrow[d, "f"] & \prod_{i \in I} F_{N_i}(K) \arrow[d, "\prod F_{N_i}(f)"] \\
K' & \prod_{i \in I} F_{N_i}(K')
\end{tikzcd}$$

Hence $F_{\bigoplus_{i \in I} N_i} \circ F_M \simeq \prod_{i \in I} F_{N_i} \circ F_M \Rightarrow F_{(\bigoplus_{i \in I} N_i) \otimes_S M} \simeq \prod F_{N_i \otimes_S M} \simeq F_{\oplus (N_i \otimes_S M)}$

which implies $(\bigoplus_{i \in I} N_i) \otimes_S M \simeq \bigoplus_{i \in I} (N_i \otimes_S M)$.

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<u>Corollary</u>. Suppose M_1 and M_2 are left S-modules. Then

$$M_1 \oplus M_2 \text{ is flat if and only if } M_1 \text{ and } M_2 \text{ are flat.}$$

<u>Pf.</u> Suppose $0 \to N \xrightarrow{f} N'$ is exact. Then

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_S M_1 \arrow[r, "id \otimes j_1"] \arrow[d, equal] \arrow[ddd, bend right, "f \otimes id_{M_1}"'] & N \otimes_S (M_1 \oplus M_2) \arrow[r, "id \otimes p"] \arrow[d] \arrow[ddd, bend left, "f \otimes id_{(M_1 \oplus M_2)}"] & N \otimes_S M_2 \arrow[r] \arrow[d, equal] \arrow[ddd, bend left, "f \otimes id_{M_2}"] & 0 \\
0 \arrow[r] & N \otimes M_1 \arrow[r, "\bar{j}_1"] \arrow[d, "f \otimes id"] & (N \otimes_S M_1) \oplus (N \otimes_S M_2) \arrow[r, "\bar{p}_2"] \arrow[d] & N \otimes_S M_2 \arrow[r] \arrow[d, "f \otimes id"] & 0 \\
0 \arrow[r] & N' \otimes M_1 \arrow[r, "\bar{j}_1"] \arrow[d, equal] & (N' \otimes_S M_1) \oplus (N' \otimes_S M_2) \arrow[r, "\bar{p}_2"] \arrow[d, "?"] & N' \otimes_S M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N' \otimes M_1 \arrow[r] & N' \otimes_S (M_1 \oplus M_2) \arrow[r, "id \otimes p"] & N' \otimes_S M_2 \arrow[r] & 0
\end{tikzcd}$$

$$n \otimes (m_1, m_2) \mapsto (n \otimes m_1, n \otimes m_2) \mapsto (f(n) \otimes m_1, f(n), m_2) \mapsto f(n) \otimes (m_1, m_2)$$

By Short Five Lemma, if $f \otimes id_{M_1}$ and $f \otimes id_{M_2}$ are injective, then $f \otimes id_{M_1 \oplus M_2}$ is injective; and so M_1, M_2 : flat $\Rightarrow M_1 \oplus M_2$: flat.

If $f \otimes id_{M_1 \oplus M_2}$ is injective, then using above diagram you can see that $f \otimes id_{M_1}$ is injective; and so $M_1 \oplus M_2$: flat $\Rightarrow M_1$: flat

And, by symmetry, $M_1 \oplus M_2$: flat $\Rightarrow M_2$: flat. ■

<u>Corollary</u>. $R^n \otimes_R M \xrightarrow{\sim} M^n$, $(r_1, \dots, r_n) \otimes m \longmapsto (r_1 m, \dots, r_n m)$.

<u>Pf.</u> $R^n \otimes_R M \xrightarrow{\sim} (R \otimes_R M)^n \xrightarrow{\sim} M^n$ (why?)

$(r_1, \dots, r_n) \otimes m \longmapsto (r_1 \otimes m, \dots, r_n \otimes m) \longmapsto (r_1 m, \dots, r_n m)$. ■

Lecture 20: Free implies projective

Friday, February 23, 2018 8:27 AM

Lemma. A free module is flat.

Pf. By $R\otimes_R M \simeq M$, $r\otimes m \mapsto m$, we deduce that R is a flat R-mod.

By induction and the previous corollary, R^n is flat for any $n\in\mathbb{Z}^+$.

Now suppose $\sum_{j=1}^n (r_i^{(j)})_{i\in I}\otimes m_j \in \ker\left(\bigoplus_{i\in I} R\otimes_R M \xrightarrow{id\otimes f} \bigoplus_{i\in I} R\otimes_R M'\right)$

where $f: M\to M'$ is an injective R-mod homomorphism.

Let $I_0 := \{i\in I \mid \exists\ j \text{ s.t. } r_i^{(j)}\neq 0\}$. Then I_0 is finite.

Notice that $\bigoplus_{i\in I} R \simeq \left(\bigoplus_{i\in I_0} R\right)\oplus\left(\bigoplus_{i\in I\setminus I_0} R\right)$. And so

$$\begin{tikzcd}
\left(\bigoplus_{i\in I_0} R\right)\otimes_R M \arrow[r, hook, "J\otimes id"] \arrow[d, "id\otimes f"'] & \left(\bigoplus_{i\in I} R\right)\otimes_R M \arrow[d, "id\otimes f"] \arrow[dl, phantom, "\circlearrowleft"] \\
\left(\bigoplus_{i\in I_0} R\right)\otimes_R M' \arrow[r, hook, "J\otimes id"'] & \left(\bigoplus_{i\in I} R\right)\otimes_R M'
\end{tikzcd}$$

And so $\left((id\otimes f)\circ(J\otimes id)\right)\left(\sum_{j=1}^n (r_i^{(j)})\otimes m_j\right) = 0.$ (*)

The above diagram together with (*) implies $id\otimes f\left(\sum_{j=1}^n (r_i^{(j)})\otimes m_j\right) = 0$

Since $|I_0|<\infty$, $\bigoplus_{i\in I_0} R$ is flat. And so

$$0\longrightarrow \left(\bigoplus_{i\in I_0} R\right)\otimes_R M \xrightarrow{id\otimes f} \left(\bigoplus_{i\in I_0} R\right)\otimes_R M \quad \text{is exact.}$$

Hence $\sum_{j=1}^n (r_i^{(j)})\otimes m_j = 0$ in $\left(\bigoplus_{i\in I_0} R\right)\otimes_R M$; and so it is zero in $\left(\bigoplus_{i\in I} R\right)\otimes_R M$. ∎

Lecture 20: Projective implies flat; tensor of algebras

Tuesday, February 20, 2018 11:43 AM

Theorem. Any projective module is flat.

Pf. Suppose P is a projective module. Then P is a direct summand of a free module F; that means $F = P \oplus N$. Since F is a flat, P is flat. $\blacksquare$

Def. Let R be a unital commutative ring, A be a (unital) ring, and $\phi: R \to A$ be a (unital) ring homomorphism. If $\operatorname{Im}(\phi) \subseteq Z(A)$, we say A is an R-algebra.

Remark. Typically we assume R is a subring of A. For instance any unital ring A is a $Z(A)$-algebra.

Remark. An R-algebra is an (R,R)-bimodule.

Theorem. Let A and B be two R-algebras. Then the following is a well-defined operation on $A \otimes_R B$; and $A \otimes_R B$ with its R-module structure and this multiplication is an R-algebra:

$$(a \otimes b) \cdot (a' \otimes b') := aa' \otimes bb'.$$

Proof was not covered during lecture

Pf. For fixed a' and b', let $f_{(a',b')}: A \times B \to A \otimes_R B$,
$f_{(a',b')}(a,b) := aa' \otimes bb'$.

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<u>R-balanced</u>. $f_{(a',b')}(ar,b) = ara'\otimes bb' = aa'r\otimes bb' = aa'\otimes rbb'$

$$= f_{(a',b')}(a, rb).$$

<u>R-linear in A</u>. $f_{(a',b')}(r_1a_1 + r_2a_2, b) = (r_1a_1+r_2a_2)a' \otimes bb'$

$$= r_1(a_1a'\otimes bb') + r_2(a_2a'\otimes bb') \checkmark$$

<u>R-linear in B</u> is similar. So $\exists$ an (R,R)-bimodule

$$\widetilde{\Psi}_{(a',b')}: A\otimes_R B \longrightarrow A\otimes_R B, \quad \widetilde{\Psi}_{(a',b')}(a\otimes b) = aa'\otimes bb'.$$

Fix $\alpha\in A\otimes_R B$, let $\phi_\alpha : A\times B \to A\otimes_R B$, $\phi_\alpha(a',b') := \widetilde{\Psi}_{(a',b')}(\alpha)$.

And so $\phi_{a\otimes b}(a',b') = aa'\otimes bb'$. Hence, if $\alpha = \sum_{i=1}^n a_i\otimes b_i$,

then $\phi_\alpha(a',b') = \widetilde{\Psi}_{(a',b')}\left(\sum_{i=1}^n a_i\otimes b_i\right)$

$$= \sum_{i=1}^n a_i a'\otimes b_i b'.$$

<u>R-balanced</u>. $\phi_\alpha(a'r, b') = \sum_{i=1}^n a_i a' r\otimes b_i b'$

$$= \sum_{i=1}^n a_i a'\otimes r b_i b' = \sum_{i=1}^n a_i a'\otimes b_i (rb')$$

$$= \phi_\alpha(a', rb').$$

Similarly one can check that ϕ_α is bi-R-linear. Hence

$\exists\ \widetilde{\Phi}_\alpha : A\otimes_R B \longrightarrow A\otimes_R B$ s.t. $\widetilde{\Phi}_\alpha(a'\otimes b') = \widetilde{\Psi}_{(a',b')}(\alpha)$

in particular $\widetilde{\Phi}_{a\otimes b}(a'\otimes b') = aa'\otimes bb'$.

Let $m : (A\otimes_R B)\times(A\otimes_R B) \longrightarrow A\otimes_R B$, $m(\alpha,\alpha') := \widetilde{\Phi}_\alpha(\alpha')$.

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Then $m(\alpha,\alpha') = \tilde{\Phi}_\alpha(\alpha')$ is R-linear in α'. And if

$\alpha' = \sum a_i' \otimes b_i'$, then $m(\alpha,\alpha') = \tilde{\Phi}_\alpha(\sum a_i' \otimes b_i')$

$$= \sum \tilde{\Phi}_\alpha(a_i' \otimes b_i')$$

$$= \sum \underbrace{\tilde{\Psi}_{(a_i',b_i')}(\alpha)}_{\text{R-linear in } \alpha}.$$

$\Rightarrow$ m is bi-R-linear.

And $m(\sum_i a_i \otimes b_i, \sum_j a_j' \otimes b_j') = \sum_{i,j} a_i a_j' \otimes b_i b_j'$.

<u>Associative</u>. $m(m(\sum_i a_i \otimes b_i, \sum_j a_j' \otimes b_j'), \sum_k a_k'' \otimes b_k'')$

$$= \sum_{i,j,k} (a_i a_j') a_k'' \otimes (b_i b_j') b_k''$$

$$\overset{\boxed{A \,\&\, B \text{ are rings}}}{=} \sum_{i,j,k} a_i (a_j' a_k'') \otimes b_i (b_j' b_k'')$$

$$= m(\sum_i a_i \otimes b_i, m(\sum_j a_j' \otimes b_j', \sum_k a_k'' \otimes b_k'')).$$

And so $\alpha \cdot \beta := m(\alpha,\beta)$ defines a multip. on $A \otimes_R B$,

and it has distribution properties and it is associative.

. Let $\phi: R \longrightarrow A \otimes_R B$, $\phi(r) := r(1 \otimes 1) = r \otimes 1 = 1 \otimes r$

Then it is easy to check that ϕ is a ring homomorphism and

$\phi(R) \subseteq Z(A \otimes_R B)$; and claim follows. ∎