

Lecture 15: The short five lemma

Friday, February 9, 2018 8:51 AM

Def. A homomorphism of short exact sequences

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

is a triple ϕ_1, ϕ_2, ϕ_3 of mod. hom. such that

the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & M_3 \rightarrow 0 \\ & & \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow \\ 0 & \rightarrow & N_1 & \rightarrow & N_2 & \rightarrow & N_3 \rightarrow 0 \end{array}$$

commutes.

• Similarly we define isomorphism of short exact sequence.

Hence we have proved a short exact sequence

$$0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0 \text{ is isomorphic to the short exact}$$

$$\text{sequence } 0 \rightarrow f_1(M_1) \rightarrow M_2 \rightarrow M_2/f_1(M_1) \rightarrow 0.$$

Lemma. (The short five lemma) Suppose (ϕ_1, ϕ_2, ϕ_3) is a

homomorphism of SES:s $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ and

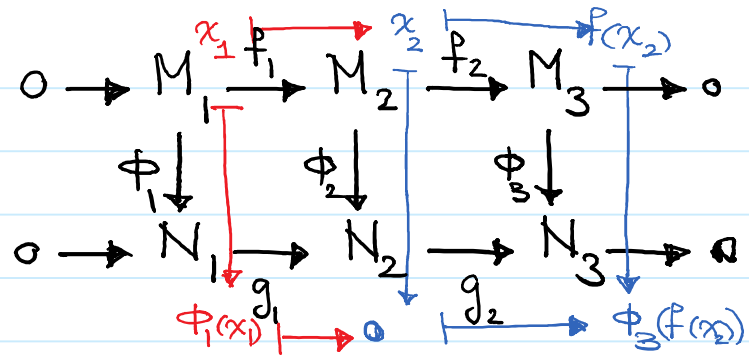
$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0. \text{ Then}$$

- If ϕ_1 and ϕ_3 are injective, then ϕ_2 is injective.
- If ϕ_1 and ϕ_3 are surjective, then ϕ_2 is surjective.
- If ϕ_1 and ϕ_3 are isomorphisms, then ϕ_2 is an isomorphism.

Lecture 15: The Short Five Lemma

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PF. (a) Suppose $\phi_2(x_2) = 0$.



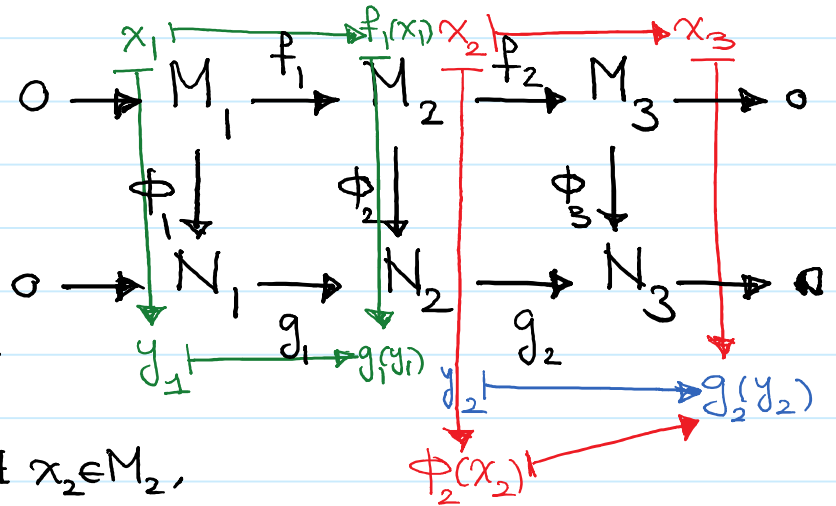
And so $\phi_3(f_2(x_2)) = g_2(\phi_2(x_2)) = 0$

Since ϕ_3 is injective, $f_2(x_2) = 0$. Hence $x_2 \in \ker f_2 = \text{Im } f_1$.

Therefore $\exists x_1 \in M_1$, $x_2 = f_1(x_1)$; and so $g_1(\phi_1(x_1)) = \phi_2(f_1(x_1)) = \phi_2(x_2) = 0$.

As g_1 and ϕ_1 are injective, $x_1 = 0$. And so $x_2 = 0$; which implies ϕ_2 is injective.

(b) Suppose $y_2 \in N_2$



Since ϕ_3 is surjective,

$\exists x_3 \in M_3$, $\phi_3(x_3) = g_2(y_2)$.

As f_2 is surjective, $\exists x_2 \in M_2$,

$f_2(x_2) = x_3$. Therefore $g_2(\phi_2(x_2)) = \phi_3(f_2(x_2)) = \phi_3(x_3) = g_2(y_2)$.

Hence $y_2 - \phi_2(x_2) \in \ker g_2 = \text{Im } g_1$. Thus $\exists y_1 \in N_1$ s.t.

$g_1(y_1) = y_2 - \phi_2(x_2)$. As ϕ_1 is surjective, $\exists x_1 \in M_1$ s.t.

$\phi_1(x_1) = y_1$. Therefore $\phi_2(f_1(x_1)) = g_1(\phi_1(x_1)) = g_1(y_1) = y_2 - \phi_2(x_2)$

And so $y_2 = \phi_2(x_2 + f_1(x_1)) \in \text{Im } \phi_2$. ■

Lecture 15: Split short exact sequences

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Proposition. Suppose $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$ is a short exact sequence of R -modules. Then the following statements are equivalent:

(a) $\exists N_2 \subseteq M_2$ submodule s.t. $M_2 = N_2 \oplus \text{Im } f_1$.

(b) $\exists g_1 \in \text{Hom}_R(M_2, M_1)$, $g_1 \circ f_1 = \text{id}_{M_1}$ $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$

(c) $\exists (\text{id}_{M_1}, \phi, \text{id}_{M_3})$: isomorphism of S.E.S.

$$\begin{array}{ccccccc} 0 & \rightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 \rightarrow 0 \\ & & \parallel & & \downarrow \phi & & \parallel \\ 0 & \rightarrow & M_1 & \rightarrow & M_1 \oplus M_3 & \rightarrow & M_3 \rightarrow 0 \end{array}$$

(d) $\exists g_2 \in \text{Hom}_R(M_3, M_2)$, $f_2 \circ g_2 = \text{id}_{M_3}$ $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$

Pf. (a) \Rightarrow (b) Since f_1 is injective, $M_1 \xrightarrow{f_1} M_2$
 $M_1 \rightarrow \text{Im } f_1, x_1 \mapsto f_1(x_1)$ is an isom. $\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_2 \\ f_1 \downarrow & \cong & \parallel \\ \text{Im } f_1 & \xrightarrow{\quad} & \text{Im } f_1 \oplus N_2 \end{array}$

So $\exists \bar{g}_1: \text{Im } f_1 \xrightarrow{\sim} M_1$ s.t. $\bar{g}_1(f_1(x_1)) = x_1$.

Let $p: M_2 \rightarrow \text{Im } f_1$ be the $\text{Im } f_1$ -component. So

$p(f_1(x_1)) = f_1(x_1)$. Let $g_1 := \bar{g}_1 \circ p: M_2 \rightarrow M_1$. Then

$$g_1(f_1(x_1)) = \bar{g}_1(p(f_1(x_1))) = \bar{g}_1(f_1(x_1)) = x_1.$$

(b) \Rightarrow (c) Let $\phi(x_2) := (g_1(x_2), f_2(x_2))$. Then

$$\phi(f_1(x_1)) = (g_1(f_1(x_1)), f_2(f_1(x_1))) = (x_1, 0) \text{ and}$$

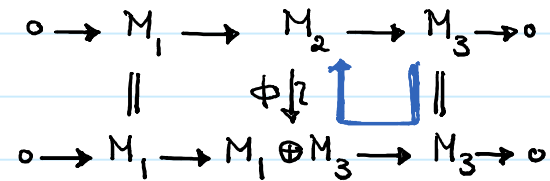
Lecture 15: Split short exact sequences

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$\phi(\phi(x_2)) = \phi(g_1(x_2), f_2(x_2)) = f_2(x_2)$. Hence $(\text{id}_{M_1}, \phi, \text{id}_{M_3})$ is a homomorphism of S.E.S.'s. Therefore, by the short five lemma, it is an isomorphism.

(c) \Rightarrow (d) Let $g_2(x_3) := \phi^{-1}(0, x_3)$.

So $\phi(g_2(x_3)) = (0, x_3)$,



which implies $f_2(g_2(x_3)) = x_3$; and so $f_2 \circ g_2 = \text{id}_{M_3}$.

(d) \Rightarrow (a) Claim. $M_2 = \ker f_2 + \text{Im } g_2$

Pf. $f_2 \circ g_2 (f_2(x_2)) = f_2(x_2) \Rightarrow g_2(f_2(x_2)) - x_2 \in \ker f_2$
 $\Rightarrow x_2 \in \ker f_2 + \text{Im } g_2$.

Claim. $\ker f_2 \cap \text{Im } g_2 = 0$

Pf. $x_2 \in \ker f_2 \cap \text{Im } g_2 \Rightarrow \left\{ \begin{array}{l} f_2(x_2) = 0 \\ x_2 = g_2(x_3) \end{array} \right\} \Rightarrow \begin{array}{l} f_2(g_2(x_3)) = 0 \\ \Rightarrow x_3 = 0 \\ \Rightarrow x_2 = 0 \end{array}$

Since $\ker f_2 = \text{Im } f_1$, we get $M_2 = \text{Im } f_1 \oplus \ker g_2$. \blacksquare

Def. If a short exact sequence satisfies the above equivalent properties, we say it splits.

Lecture 15: Basics of Category theory

Monday, February 12, 2018 12:30 AM

We very briefly discuss what a category is:

A category \mathcal{C} has:

- class $\text{obj}(\mathcal{C})$ of objects
- class $\text{hom}(\mathcal{C})$ of morphisms

($\forall a, b \in \mathcal{C}, \text{hom}_{\mathcal{C}}(a, b).$)

$$\begin{array}{c} a \xrightarrow{f} b \xrightarrow{g} c \\ \xrightarrow{g \circ f} \end{array}$$

Ex. Set :• $\text{Obj}(\text{Set})$: sets

- $\text{hom}_{\text{Set}}(A, B)$: all the functions from A to B
- Grp :• $\text{Obj}(\text{Grp})$: groups
 - $\text{hom}_{\text{Grp}}(G_1, G_2)$: group homomorphisms.
- Ab :• $\text{Obj}(\text{Ab})$: Abelian groups
 - $\text{hom}_{\text{Ab}}(G_1, G_2)$: group hom.
- R-mod :• $\text{Obj}(\text{R-mod})$: R-modules
 - $\text{hom}_{\text{R-mod}}(M_1, M_2) := \text{Hom}_{\text{R}}(M_1, M_2).$

Def. A functor F from a category \mathcal{C} to a category \mathcal{D}

gives us

$$\begin{array}{ccc} \text{obj}(\mathcal{C}) & & \text{obj}(\mathcal{D}) \\ a & & F(a) \\ \downarrow \phi & \rightsquigarrow & \downarrow F(\phi) \\ b & & F(b) \end{array} \quad \begin{array}{l} \text{and } F(\phi_1 \circ \phi_2) \\ = F(\phi_1) \circ F(\phi_2). \\ \text{and } F(\text{id}_a) = \text{id}_{F(a)}. \end{array}$$

Lecture 15: Representable functor

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Ex. (Forgetful Functor) $F: \text{Grp} \rightarrow \text{Set}$

$$\left. \begin{array}{l} F(G) := G \\ F(\phi) := \phi \end{array} \right\} \begin{array}{l} \text{just "forget" the group} \\ \text{structure of } G. \end{array}$$

$$F: \text{Ab} \rightarrow \text{Grp}$$

"forget" that G is Abelian

$$F: \mathbb{R}\text{-mod} \rightarrow \text{Ab}$$

"forget" that M has scalar multiplication