Lecture 15: The short five lemma

Friday, February 9, 2018 8:51 AM

Def. A homomorphism of short exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0 \quad \text{and} \quad 0 \to N_1 \to N_2 \to N_3 \to 0$$

is a triple ϕ_1, ϕ_2, ϕ_3 of mod. hom. such that

the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\phi_1"] & M_2 \arrow[r] \arrow[d, "\phi_2"] & M_3 \arrow[r] \arrow[d, "\phi_3"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0 \end{tikzcd}$$

- Similarly we define isomorphism of short exact sequence.

Hence we have proved a short exact sequence

$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is isomorphic to the short exact

sequence $\quad 0 \to f_1(M_1) \to M_2 \to M_2/f_1(M_1) \to 0$.

Lemma. (The short five lemma) Suppose (ϕ_1, ϕ_2, ϕ_3) is a homomorphism of SES·s $\quad 0 \to M_1 \to M_2 \to M_3 \to 0$ and $0 \to N_1 \to N_2 \to N_3 \to 0$. Then

(a) If ϕ_1 and ϕ_3 are injective, then ϕ_2 is injective.

(b) If ϕ_1 and ϕ_3 are surjective, then ϕ_2 is surjective.

(c) If ϕ_1 and ϕ_3 are isomorphisms, then ϕ_2 is an isomorphism.

Lecture 15: The Short Five Lemma

Sunday, February 11, 2018 10:20 PM

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & M_3 \arrow[r] \arrow[d, "\phi_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r] & 0
\end{tikzcd}$$

<u>Pf.</u> (a) Suppose $\phi_2(x_2)=0$.

And so $\phi_3(f(x_2)) = g_2(\phi_2(x))$
$= 0$

Since ϕ_3 is injective, $f(x_2)=0$. Hence $x_2 \in \ker f_2 = \operatorname{Im} f_1$.

Therefore $\exists\, x_1 \in M_1$, $x_2 = f_1(x_1)$; and so $g_1(\phi_1(x_1)) = \phi_2(f_1(x_1))$
$= \phi_2(x_2) = 0$.

As g_1 and ϕ_1 are injective, $x_1 = 0$. And so $x_2 = 0$; which implies ϕ_2 is injective.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & M_3 \arrow[r] \arrow[d, "\phi_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r] & 0
\end{tikzcd}$$

(b) Suppose $y_2 \in N_2$

Since ϕ_3 is surjective,

$\exists\, x_3 \in M_3$, $\phi_3(x_3) = g_2(y_2)$.

As f_2 is surjective, $\exists\, x_2 \in M_2$,

$f_2(x_2) = x_3$. Therefore $g_2(\phi_2(x_2)) = \phi_3(f_2(x_2)) = \phi_3(x_3) = g_2(y_2)$.

Hence $y_2 - \phi_2(x_2) \in \ker g_2 = \operatorname{Im} g_1$. Thus $\exists\, y_1 \in N_1$ s.t.

$g_1(y_1) = y_2 - \phi_2(x_2)$. As ϕ_1 is surjective, $\exists\, x_1 \in M_1$ s.t.

$\phi_1(x_1) = y_1$. Therefore $\phi_2(f_1(x_1)) = g_1(\phi_1(x_1)) = g_1(y_1) = y_2 - \phi_2(x_2)$

And so $y_2 = \phi_2(x_2 + f_1(x_1)) \in \operatorname{Im} \phi_2$. ■

<u>Proposition</u>. Suppose $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a short exact sequence of R-modules. Then the following statements are equivalent:

(a) $\exists\, N_2 \subseteq M_2$ submodule s.t. $M_2 = N_2 \oplus \operatorname{Im} f_1$.

(b) $\exists\, g_1 \in \operatorname{Hom}_R(M_2, M_1)$, $g_1 \circ f_1 = \mathrm{id}_{M_1}$

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, shift left, "f_1"] & M_2 \arrow[l, shift left, "g_1"] \arrow[r, "f_2"] & M_3 \arrow[r] & 0 \end{tikzcd}$$

(c) $\exists\, (\mathrm{id}_{M_1}, \Phi, \mathrm{id}_{M_3})$: isomorphism of S.E.S.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, equal] & M_2 \arrow[r, "f_2"] \arrow[d, "\Phi", "\wr"'] & M_3 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_1 \oplus M_3 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

(d) $\exists\, g_2 \in \operatorname{Hom}_R(M_3, M_2)$, $f_2 \circ g_2 = \mathrm{id}_{M_3}$

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, shift left, "f_2"] & M_3 \arrow[l, shift left, "g_2"] \arrow[r] & 0 \end{tikzcd}$$

<u>Pf</u>. (a) $\Rightarrow$ (b) Since f_1 is injective,

$M_1 \to \operatorname{Im} f_1$, $x_1 \mapsto f_1(x_1)$ is an isom.

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\circlearrowright"] & M_2 \arrow[d, equal] \\ \operatorname{Im} f_1 & \operatorname{Im} f_1 \oplus N_2 \arrow[l] \end{tikzcd}$$

So $\exists\, \overline{g_1} : \operatorname{Im} f_1 \xrightarrow{\sim} M_1$ s.t. $\overline{g_1}(f_1(x_1)) = x_1$.

Let $p : M_2 \to \operatorname{Im} f_1$ be the $\operatorname{Im} f_1$-component. So

$p(f_1(x_1)) = f_1(x_1)$. Let $g_1 := \overline{g_1} \circ p : M_2 \to M_1$. Then

$$g_1(f_1(x_1)) = \overline{g_1}(p(f_1(x_1))) = \overline{g_1}(f_1(x_1)) = x_1.$$

(b) $\Rightarrow$ (c) Let $\Phi(x_2) := (g_1(x_2), f_2(x_2))$. Then

$\Phi(f_1(x_1)) = (g_1(f_1(x_1)), f_2(f_1(x_2))) = (x_1, 0)$ and

$p(\phi(x_2)) = p(g_1(x_2), f_2(x_2)) = f_2(x_2)$. Hence $(id_{M_1}, \phi, id_{M_3})$ is a homomorphism of S.E.S.'s. Therefore, by the short five lemma, it is an isomorphism.

(c) $\Rightarrow$ (d) Let $g_2(x_3) := \phi^{-1}(0, x_3)$.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & M_2 \arrow[r] \arrow[d, "\phi", "\wr"'] & M_3 \arrow[r] \arrow[d, equal] \arrow[l, bend right, "g_2"'] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_1 \oplus M_3 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

So $\phi(g_2(x_3)) = (0, x_3)$, which implies $f_2(g_2(x_3)) = x_3$; and so $f_2 \circ g_2 = id_{M_3}$.

(d) $\Rightarrow$ (a) Claim. $M_2 = \ker f_2 + \operatorname{Im} g_2$

Pf. $f_2 \circ g_2(f_2(x_2)) = f_2(x_2) \Rightarrow g_2(f_2(x_2)) - x_2 \in \ker f_2$
$\Rightarrow x_2 \in \ker f_2 + \operatorname{Im} g_2$.

Claim. $\ker f_2 \cap \operatorname{Im} g_2 = 0$

Pf. $x_2 \in \ker f_2 \cap \operatorname{Im} g_2 \Rightarrow \left\{ \begin{array}{l} f_2(x_2) = 0 \\ x_2 = g_2(x_3) \end{array} \right\} \Rightarrow f_2(g_2(x_3)) = 0$
$\Rightarrow x_3 = 0$
$\Rightarrow x_2 = 0$

Since $\ker f_2 = \operatorname{Im} f_1$, we get $M_2 = \operatorname{Im} f_1 \oplus \ker g_2$. ∎

<u>Def</u>. If a short exact sequence satisfies the above equivalent properties, we say it <u>splits</u>.

Lecture 15: Basics of Category theory

Monday, February 12, 2018 12:30 AM

We very briefly discuss what a category is:

A category $\mathcal{C}$ has
- class $\mathrm{obj}(\mathcal{C})$ of <u>objects</u>
- class $\hom(\mathcal{C})$ of <u>morphisms</u>

$(\forall a, b \in \mathcal{C},\ \hom_{\mathcal{C}}(a,b).)$

$$a \xrightarrow{f} b \xrightarrow{g} c, \qquad a \xrightarrow{g\circ f} c$$

<u>Ex.</u>
- Set:
  - $\mathrm{Obj}(\mathrm{Set})$: sets
  - $\hom_{\mathrm{set}}(A,B)$: all the functions from A to B
- Grp:
  - $\mathrm{Obj}(\mathrm{Grp})$: groups
  - $\hom_{\mathrm{Grp}}(G_1,G_2)$: group homomorphisms.
- Ab:
  - $\mathrm{Obj}(\mathrm{Ab})$: Abelian groups
  - $\hom_{\mathrm{Ab}}(G_1,G_2)$: group hom.
- R-mod:
  - $\mathrm{Obj}(R\text{-mod})$: R-modules
  - $\hom_{R\text{-mod}}(M_1,M_2) := \mathrm{Hom}_R(M_1,M_2)$.

<u>Def</u>. A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ gives us

$$\begin{array}{ccc} \mathrm{obj}(\mathcal{C}) & & \mathrm{obj}(\mathcal{D}) \\ a & & F(a) \\ \downarrow \phi & \longmapsto & \downarrow F(\phi) \\ b & & F(b) \end{array}$$

and $F(\phi_1 \circ \phi_2) = F(\phi_1)\circ F(\phi_2)$. and $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

Lecture 15: Representable functor

Monday, February 12, 2018 12:56 AM

Ex. (Forgetful functor) $F: Grp \to Set$

$F(G) := G$
$F(\phi) := \phi$
} just "forget" the group structure of G.

$F: Ab \to Grp$

"forget" that G is Abelian

$F: R\text{-mod} \to Ab$

"forget" that M has scalar multiplication