# Lecture 11: Submodules of free modules over a PID

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In the previous lecture we were proving the following theorem:

**Theorem.** Let $D$ be a PID, and $M$ be a submod. of $D^n$.

Then (a) $M$ is a free module.

(b) $\exists\, x_1, \ldots, x_n \in D^n$ and $a_1, \ldots, a_m \in D \setminus \{0\}$ s.t.

(b-1) $a_1 | a_2 | \cdots | a_m$.

(b-2) $D^n = \bigoplus_{i=1}^{n} D x_i$.

(b-3) $M = \bigoplus_{i=1}^{m} D a_i x_i$.

**Recall.** • For any $\phi \in \mathrm{Hom}_D(D^n, D)$, $\exists\, a_\phi \in D$ s.t. $\phi(M) = \langle a_\phi \rangle$.

• Let $\Sigma := \{\phi(M) \mid \phi \in \mathrm{Hom}_D(D^n, D)\}$. Then it has a maximal element. Suppose $\phi_1(M) = \langle a_1 \rangle$ is a maximal element of $\Sigma$ and $\phi_1(y_1) = a_1$ for some $y_1 \in M$. Then we proved

**Claim 1.** $\forall \phi \in \mathrm{Hom}_D(D^n, D)$, $a_1 | \phi(y_1)$.

**Corollary of claim 1.** $y_1 = a_1 x_1$ for some $x_1 \in D^n$ and $\phi_1(x_1) = 1$.

**Claim 2.** $D^n = \ker \phi_1 \oplus D x_1$ and $M = (\ker \phi_1 \cap M) \oplus D a_1 x_1$.

Now by induction on $\mathrm{rank}(M)$, we prove that $M$ is free.

Claim 3. $M$ is free.

Pf of claim 3. We proceed by strong induction on $\mathrm{rank}(M)$.

If $M=0$, there is nothing to prove. By Claim 2,

$$\mathrm{rank}(M \cap \ker \phi_1) \leq \mathrm{rank}(M) - 1 .$$

Hence by the strong induction hypothesis, $M \cap \ker \phi_1$ is a free $D$-mod. Hence, by Claim 2,

$$M = (M \cap \ker \phi_1) \oplus D a_1 x_1 \simeq D^r \oplus D,$$

is a free module.

Claim 4. Existence of $x_i$'s and $a_i$'s.

Pf of claim 4. We proceed by strong induction on $\mathrm{rank}(D^n) = n$. By Claim 3, $\ker \phi_1$ is a free module; and by claim 2, $\mathrm{rank}(\ker \phi_1) = n-1$. Hence, by the induction hypothesis, there are $x_2, \ldots, x_n \in \ker \phi_1$ and $a_2, \ldots, a_m \in D \setminus \{0\}$ s.t.

- $a_2 | a_3 | \cdots | a_m$
- $\ker \phi_1 = \bigoplus_{i=2}^{n} D x_i$
- $\ker \phi_1 \cap M = \bigoplus_{i=2}^{m} D a_i x_i$ .

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Hence, by Claim 2, $D^n = \bigoplus_{i=1}^{n} D x_i$, $M = \bigoplus_{i=1}^{m} D a_i x_i$.

So it is enough to prove $a_1 | a_2$.

Let $\phi(x_1) = \phi(x_2) = 1$ and $\phi(x_3) = \cdots = \phi(x_n) = 0$;

and extend it linearly to a D-mod. homomorphism from

$D^n = \bigoplus_{i=1}^{n} D x_i$ to $D$. Hence

$$\phi(a_1 x_1), \phi(a_2 x_2) \in \phi(M) \Rightarrow a_1, a_2 \in \phi(M)$$

$$\left.\begin{array}{l} \Rightarrow \langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \phi(M) \\ \text{Since } \langle a_1 \rangle \text{ is maximal in } \Sigma \\ \text{and } \phi(M) \in \Sigma \end{array}\right\} \Rightarrow \langle a_1 \rangle = \phi(M)$$

$$\Rightarrow a_2 \in \langle a_1 \rangle$$

$$\Rightarrow a_1 | a_2.$$

∎

<u>Fundamental theorem of f.g. mod. over a PID.</u> (Existence)

Let $D$ be a PID and $M$ be a f.g. D-mod.

Then $\exists\, a_1 | a_2 | \cdots | a_m$ s.t. $M \simeq D^r \oplus \bigoplus_{i=1}^{m} D/\langle a_i \rangle$.

<u>Pf.</u> Suppose $M = Dm_1 + Dm_2 + \cdots + Dm_n$. Let $\phi: D^n \to M$ be

the D-mod. hom. given by $\phi(e_i) := m_i$. Then $\phi$ is surjective.

Hence, by the 1st isomorphism theorem, $M \simeq D^n / \ker \phi$.

$\ker \phi$ is a submod. of $D^n$. Hence by the previous theorem

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$\exists\, x_1, \dots, x_n \in D^n$, $a_1, \dots, a_m \in D \setminus \{0\}$ s.t.

- $a_1 \mid a_2 \mid \cdots \mid a_m$
- $D^n = \bigoplus_{i=1}^n Dx_i$
- $\ker \phi = \bigoplus_{i=1}^m Da_i x_i$

$\Big\} \Rightarrow D^n / \ker \phi \simeq D^{n-m} \oplus \bigoplus_{i=1}^m D/\langle a_i \rangle$.

(why?)

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**Fundmental theorem of f.g. mod. over a PID (Uniqueness)**

Suppose $a_1 | \cdots | a_m$, $b_1 | \cdots | b_s$, $\quad (a_i, b_j \in D \setminus \{0\})$

$$D^n \oplus \bigoplus_{i=1}^m D/\langle a_i \rangle \simeq D^r \oplus \bigoplus_{i=1}^s D/\langle b_i \rangle .$$

Then $n = r$, $m = s$, and $\langle a_i \rangle = \langle b_i \rangle$ for any $i$.

Pf. Claim 1. $\mathrm{rank}\,(\underbrace{D^n \oplus \bigoplus_{i=1}^m D/\langle a_i \rangle}_{M}) = n$.

Pf. $v_1, \dots, v_{n+1} \in M \Rightarrow a_m v_1, \dots, a_m v_{n+1} \in D^n$

$\Rightarrow a_m v_1, \dots, a_m v_{n+1}$ are linearly depen.

$\Rightarrow v_1, \dots, v_{n+1}$ are linearly dependent; and claim follows.

Corollary of claim 1. $n = r$.

Claim 2. $\mathrm{Tor}(M) := \{ x \in M \mid \exists\, a \in D \setminus \{0\},\ ax = 0 \} = \bigoplus_{i=1}^m D/\langle a_i \rangle$.

Pf of claim 2. Since $a_m \left( \bigoplus_{i=1}^m D/\langle a_i \rangle \right) = 0$, RHS $\subseteq$ LHS.

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For $x \in \mathrm{Tor}(M)$, $x = x_f + x_t$ where $x_f \in D^n$, $x_t \in \bigoplus_{i=1}^{m} D/\langle a_i \rangle$;

and $ax = 0$ for some $a \in D \setminus \{0\}$. Hence $a x_f + a x_t = 0$;

which implies $a x_f = 0$; and so $x_f = 0$; therefore $x = x_t \in$ RHS.

<u>Corollary of claim 2.</u> $\bigoplus_{i=1}^{m} D/\langle a_i \rangle \simeq \bigoplus_{i=1}^{s} D/\langle b_i \rangle$.

- To show $\langle a_i \rangle = \langle b_i \rangle$, we have to show for any irreducible element $p$ of $D$, $v_p(a_i) = v_p(b_i)$ (Recall that $v_p(a)$ is the power of $p$ in the decomposition of $a$ to irreducibles.)

In $\mathbb{Z}$, if we are only interested in powers of 2, then we can go to a larger ring where all the odd numbers have inverse; let $S = \mathbb{Z} \setminus 2\mathbb{Z}$, and consider $S^{-1}\mathbb{Z}$. Then

$\langle a \rangle = \langle 2^{v_2(a)} \rangle$ in $S^{-1}\mathbb{Z}$. For a prime in a PID we can do the same: $S_p := D \setminus pD$, we go to $S_p^{-1} D$

and get $\langle a \rangle = \langle p^{v_p(a)} \underbrace{\cdots}_{\text{are units in } S_p^{-1}D} \rangle = \langle p^{v_p(a)} \rangle$.

(Here we are using: $\mathrm{Spec}(D) = \mathrm{Max}(D) \cup \{0\}$.)