# Lecture 08: Parametrizing module structures on M

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For a group $G$ and a set $X$, we saw that there is a bijection between $\{m: G\times X \to X \mid \text{action}\}$ and $\mathrm{Hom}(G, S_X)$.

Now for a given ring $R$ and an abelian group $M$, we'd like to do the same; this means we would like to parametrize all the possible scalar multiplications $m: R\times M \to M$ which makes $M$ into a left $R$-mod.

| | |
|---|---|
| Group: $G$ | Ring : $R$ |
| Set : $X$ | Abelian gp : $M$ |
| Action: $m: G\times X \to X$ | Mod. : $m: R\times M \to M$ |
| $S_X$ (group) | ? (ring) |
| $\underset{gp}{\mathrm{Hom}}(G, S_X)$ | $\underset{\substack{\text{ring} \\ (\text{unital})}}{\mathrm{Hom}}(R, ?)$ |

<u>Proposition</u>. $\mathrm{End}(M) := \{\theta: M \to M \mid \text{gp homomorphism}\}$ is a ring w.r.t. $\begin{cases} (\theta_1+\theta_2)(m) := \theta_1(m)+\theta_2(m) \\ (\theta_1\cdot\theta_2)(m) := \theta_1(\theta_2(m)). \end{cases}$

# Lecture 08: Endomorphisms of an abelian group

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Pf. ① $(\theta_1+\theta_2)(m-m') = \theta_1(m-m')+\theta_2(m-m')$

$= (\theta_1(m)+\theta_2(m)) - (\theta_1(m')+\theta_2(m'))$

$= (\theta_1+\theta_2)(m) - (\theta_1+\theta_2)(m')$

So $\theta_1+\theta_2 \in \mathrm{End}(M)$.

② $(\theta_1\cdot\theta_2)(m-m') = \theta_1(\theta_2(m-m'))$

$= \theta_1(\theta_2(m)-\theta_2(m'))$

$= \theta_1(\theta_2(m)) - \theta_1(\theta_2(m'))$

$= (\theta_1\cdot\theta_2)(m) - (\theta_1\cdot\theta_2)(m')$.

③ Associativity is clear.

④ Distribution. $(\theta_1\cdot(\theta_2+\theta_3))(m) = \theta_1((\theta_2+\theta_3)(m))$

$= \theta_1(\theta_2(m)+\theta_3(m)) = \theta_1(\theta_2(m))+\theta_1(\theta_3(m))$

$= (\theta_1\cdot\theta_2)(m)+(\theta_1\cdot\theta_3)(m)$

$= (\theta_1\cdot\theta_2+\theta_1\cdot\theta_3)(m)$.

And $((\theta_1+\theta_2)\cdot\theta_3)(m) = (\theta_1+\theta_2)(\theta_3(m))$

$= \theta_1(\theta_3(m))+\theta_2(\theta_3(m))$

$= (\theta_1\cdot\theta_3)(m)+(\theta_2\cdot\theta_3)(m)$

$= (\theta_1\cdot\theta_3+\theta_2\cdot\theta_3)(m)$. ∎

# Lecture 08: Scalar multiplication and endomorphisms

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<u>Theorem.</u> Let $R$ be a unital ring and $M$ be an abelian gp. Then the following maps are inverse of each other:

$$\left\{ \underline{m}: R\times M \to M \;\middle|\; \begin{array}{l}\text{defines a left}\\ R\text{-mod.}\end{array} \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \underbrace{\mathrm{Hom}}_{\text{unital rings}}(R, \mathrm{End}(M))$$

$$\underbrace{\underbrace{\big(\underbrace{\Phi(\underline{m})}_{\text{in } \mathrm{Hom}(R,\mathrm{End}(M))}\big)(r)}_{\text{in } \mathrm{End}(M)}(x)}_{\text{in } M} := \underbrace{\underline{m}(r,x)}_{\text{scalar multi. in the module.}}$$

$$\underbrace{\Psi(\underset{\text{in}}{\theta})}_{\text{a scalar multi.}}(r,m) := (\underbrace{\theta(r)}_{\text{in } \mathrm{End}(M)})(m)$$

<u>The main points of proof.</u>

- Suppose $M$ is an $R$-mod. Then, for any $r\in R$,

  $l_r: M \to M$, $l_r(m) := r.m$ is an abelian group homomorphism. So $l_r \in \mathrm{End}(M)$

- $R \to \mathrm{End}(M)$ is a ring homomorphism.
  $r \mapsto l_r$

$$l_{r_1+r_2}(m) = (r_1+r_2)\cdot m = r_1\cdot m + r_2\cdot m = l_{r_1}(m) + l_{r_2}(m)$$
$$= (l_{r_1}+l_{r_2})(m).$$

$$l_{r_1r_2}(m) = (r_1r_2).m = r_1.(r_2.m) = l_{r_1}(l_{r_2}(m)) = (l_{r_1}\cdot l_{r_2})(m).$$

- If $\theta: R \to \mathrm{End}(M)$ is a ring homomorphism, then
  (which sends $1 \mapsto \mathrm{id}_M$.)

# Lecture 08: Endomorphisms and module structure

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$r.m := (\theta(r))(m)$ satisfies properties of scalar multiplica. in modules:

$$(r_1+r_2)\cdot m = \big(\theta(r_1+r_2)\big)(m) = (\theta(r_1)+\theta(r_2))(m)$$

$$= \theta(r_1)(m) + \theta(r_2)(m) = r_1\cdot m + r_2\cdot m$$

$$r.(m_1+m_2) = (\theta(r))(m_1+m_2) = (\theta(r))(m_1) + (\theta(r))(m_2)$$

$$= r\cdot m_1 + r.m_2.$$

$$1.m = (\theta(1))(m) = id_M(m) = m. \qquad \blacksquare$$

<u>Proposition</u>. Suppose $M$ is a left $R$-mod., and $\theta: R \to End(M)$ is the induced unital ring homomorphism. Then

$$End_R(M) := \{f: M \to M \mid R\text{-mod homomorphism}\}$$

is equal to $C_{End(M)}(\theta(R))$; in particular, it is a subring; and, if $R$ is commutative, then $\theta(R) \subseteq Z(End_R(M))$.

<u>Pf</u>. $f \in End_R(M) \iff \begin{cases} f \in End(M) \\ \forall r \in R, \forall m \in M, \; f(r.m) = r.f(m) \end{cases}$

$$\iff \begin{cases} f \in End\, M, \\ f \circ \theta(r) = \theta(r) \circ f \end{cases} \iff f \in C_{End(M)}(\theta(R)).$$

$\blacksquare$

# Lecture 08: Generating set

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**Def/Lemma.** Suppose $M$ is a left $R$-mod., and $A \subseteq M$. The submodule generated by $A$ is the smallest submodule of $M$ that contains $A$. It is denoted by $RA$. Then $RA$ exists and

$$RA = \bigcap_{\substack{N \subseteq M \\ \text{submod.} \\ A \subseteq N}} N \quad \text{and} \quad RA = \left\{ \sum_{i=1}^{m} r_i a_i \,\middle|\, r_i \in R, a_i \in A \right\}.$$

**Pf.** • If $\{N_i\}_{i\in I}$ is a family of submod, then $\bigcap_{i \in I} N_i$ is also a submod. (why?). And so $\bigcap_{\substack{N \subseteq M \\ \text{submod} \\ A \subseteq N}} N$ is the smallest submod of $M$ that contains $A$.

• $\forall r_i \in R, a_i \in A \Rightarrow \sum r_i a_i \in RA$.

So The RHS $\subseteq RA$.

• $1.a = a \Rightarrow A \subseteq$ RHS. So it is enough to observe that RHS is a submod. ∎

**Def.** Suppose $\{N_i\}_{i \in I}$ is a family of submodules of $M$. Then we let $\sum_{i \in I} N_i := \left\{ \sum_{i \in I} n_i \,\middle|\, n_i \in N_i \text{ where } n_i\text{'s are zero except for finitely many } i \right\}$.

# Lecture 08: Summation of modules; cyclic modules

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<u>Lemma.</u> Suppose $\{N_i\}_{i\in I}$ is a family of submod. of $M$.

Then $\sum_{i\in I} N_i$ is the smallest submod. of $M$ that contains

$N_i$'s as subsets.

<u>Pf.</u> $\sum_{i\in I} n_i + \sum_{i\in I} n_i' = \sum_{i\in I} \underbrace{n_i+n_i'}_{\text{in } N_i}$ ; $\Rightarrow$ it is a submod.

$r\sum_{i\in I} n_i = \sum_{i\in I} r n_i$ .

By the previous lemma, the mod. gen. by $\bigcup_{i\in I} N_i$ contains $\sum_{i\in I} N_i$ ; and so they are equal. ∎