Lecture 06: Hilbert's basis theorem

Monday, January 22, 2018 6:31 AM

<u>Theorem</u>. Suppose A is a Noetherian unital commutative ring. Then $A[x]$ is Noetherian.

<u>Pf</u>. (Cont.) Suppose $\mathfrak{a} \lhd A[x]$. Let

$$ld(\mathfrak{a}) := \{a \in A \mid a \text{ is the leading coeff. of an element of } \mathfrak{a}\} \cup \{0\}$$ and

$$ld_m(\mathfrak{a}) := \{a \in A \mid a \text{ is the leading coeff. of an element of deg. } m \text{ of } \mathfrak{a}\} \cup \{0\}.$$

We showed $ld(\mathfrak{a})$ and $ld_m(\mathfrak{a}) \lhd A$. And so they are f.g.; say $ld(\mathfrak{a}) = \langle a_1, \ldots, a_n \rangle$ and

$$ld_m(\mathfrak{a}) = \langle b_{1m}, \ldots, b_{n_m m} \rangle;$$

and $f_i = a_i x^{d_i} + \cdots \in \mathfrak{a}$ and

$$g_{im} = b_{im} x^m + \cdots \in \mathfrak{a}.$$

<u>Claim.</u> $\mathfrak{a} = \langle f_i, g_{jm} \mid 1 \leq i \leq n, 1 \leq m \leq \max_{l=1}^{n} d_l, 1 \leq j \leq n_m \rangle$

<u>Pf</u> Let $\mathfrak{a}'$ be the RHS. So it is clear that $\mathfrak{a} \supseteq \mathfrak{a}'$.

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By strong induction on $\underline{\deg f}$ we show, if $f \in \mathfrak{a}$, then $f \in \mathfrak{a}'$.

$f(x) = a x^d + \cdots \in \mathfrak{a}$.

<u>Case 1.</u> $d = \deg f > \max\{d_1, \ldots, d_n\}$.

$\Rightarrow a \in \mathrm{ld}(\mathfrak{a}) \Rightarrow a = r_1 a_1 + \cdots + r_n a_n$

$\Rightarrow a x^d = (r_1 x^{d-d_1})(a_1 x^{d_1}) + \cdots + (r_n x^{d-d_n})(a_n x^{d_n})$

$\Rightarrow \begin{cases} f(x) - \sum r_i x^{d-d_i} f_i(x) \in \mathfrak{a} \\ \deg(f(x) - \sum r_i x^{d-d_i} f_i(x)) < \deg f = d. \end{cases}$

$\Rightarrow$ By the strong induction hypothesis

$f(x) - \underbrace{\sum r_i x^{d-d_i} f_i(x)}_{\text{in } \mathfrak{a}'} \in \mathfrak{a}'$; this implies $f(x) \in \mathfrak{a}'$.

<u>Case 2.</u> $d \le \max\{d_1, \ldots, d_n\}$.

$\Rightarrow a \in \mathrm{ld}_d(\mathfrak{a}) \Rightarrow a = r_1 b_{1d} + \cdots + r_{n_d} b_{n_d d}$.

$\Rightarrow a x^d = r_1 b_{1d} x^d + \cdots + r_{n_d} b_{n_d d} x^d$

$\Rightarrow \begin{cases} f(x) - \sum r_i g_{id} \in \mathfrak{a} \\ \deg(f(x) - \sum r_i g_{id}) < d = \deg f \end{cases}$

$\Rightarrow$ By the strong induction hypothesis

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$$f(x) - \underbrace{\sum r_i g_{id}(x)}_{\text{in } \mathfrak{a}'} \in \mathfrak{a}' \text{ ; this implies } f(x) \in \mathfrak{a}'. \quad \blacksquare$$

You have seen how to define the field of fractions of an integral domain in your previous algebra courses. We more or less follow the same method to define $S^{-1}A$.

Suppose A is a commutative unital ring, and $S \subseteq A$ is a multiplicatively closed subset. Now we are going to construct the ring of fractions of A with respect to S.

On $A \times S$ consider the following relation:

$$(a_1, s_1) \sim (a_2, s_2) \Leftrightarrow \exists\, s \in S, \quad s(a_1 s_2 - a_2 s_1) = 0.$$

① $(a,s) \sim (a,s)$. $1(as - as) = 0$.

② $(a_1, s_1) \sim (a_2, s_2) \Rightarrow (a_2, s_2) \sim (a_1, s_1)$ it is clear

③ $\left.\begin{array}{l}(a_1,s_1)\sim(a_2,s_2)\\(a_2,s_2)\sim(a_3,s_3)\end{array}\right\} \Rightarrow \left\{\begin{array}{l} s(a_1 s_2 - a_2 s_1) = 0 \\ s'(a_2 s_3 - a_3 s_2) = 0\end{array}\right. \Rightarrow \left\{\begin{array}{l} a_1 s s_2 = a_2 s s_1 \\ a_2 s' s_3 = a_3 s' s_2 \end{array}\right\}$

$$\Rightarrow a_1 s s' s_2 s_3 = a_3 s s' s_1 s_2$$

$$\Rightarrow \underbrace{s s' s_2}_{\text{in } S} (a_1 s_3 - a_3 s_1) = 0 \Rightarrow (a_1, s_1) \sim (a_3, s_3).$$

Lecture 06: Ring of fractions with respect to S

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So $\sim$ is an equiv. relation on $A\times S$. $[(a,s)]$ is denoted by a/s.

And we let $S^{-1}A := \{a/s \mid a\in A, s\in S\}$.

We define $a/s + b/r := ar+bs/rs$ and $a/s \cdot b/r := ab/rs$;

<u>Exercise</u> • Check that the above operations are well-defined.

• Check that $S^{-1}A$ is a ring.

<u>Obser. 1</u>. $\phi: A \to S^{-1}A$, $\phi(a) := \frac{a}{1}$ is a ring homomorphism.

<u>Pf.</u> $\phi(a)+\phi(b) = a/1 + b/1 = \frac{a+b}{1} = \phi(a+b)$.

$\phi(a)\cdot\phi(b) = \frac{a}{1}\cdot\frac{b}{1} = \frac{ab}{1} = \phi(ab)$.

<u>Obser. 2</u>. $0\in S \Rightarrow S^{-1}A = 0$

<u>Pf.</u> $\forall\, a\in A, s\in S$, $\underbrace{0}_{\text{in } S}\times(s\times 0 - a\times 1) = 0$

And so $\frac{a}{s} = \frac{0}{1}$. ∎

<u>Observ. 3</u> $\phi: A \to S^{-1}A$ is injective $\Leftrightarrow$ S does not contain a zero-divisor.

<u>Pf.</u> $(\Rightarrow)$ $sa = 0 \Rightarrow \frac{sa}{s} = \frac{0}{1} \Rightarrow \frac{a}{1} = \frac{0}{1}$

$\Rightarrow \phi(a) = 0 \Rightarrow a = 0$.

$(\Leftarrow)$ $\phi(a) = 0 \Rightarrow \frac{a}{1} = \frac{0}{1} \Rightarrow \exists\, s\in S,\ s(a-0) = 0$

$\Rightarrow sa = 0 \Rightarrow a = 0$. ∎

Lecture 06: Universal property of ring of fractions

Thursday, January 11, 2018 11:58 PM

<u>Observat. 4</u>. $\mathfrak{p} \in \operatorname{Spec}(A) \Rightarrow S_{\mathfrak{p}} := A \setminus \mathfrak{p}$ is a multiplicatively closed set. $S_{\mathfrak{p}}^{-1}A$ is called the <u>localization</u> of A at $\mathfrak{p}$ and it is denoted by $A_{\mathfrak{p}}$.

<u>Obser. 5</u>. Suppose D is an integral domain. Then 0 is a prime ideal and the localization of D at 0 is the field of fractions of D.

<u>Universal Property of the ring of fractions</u>

Suppose A is a unital commutative ring, and $S \subseteq A$ is a multiplicatively closed subset. Suppose B is a unital commutative ring, and $\psi : A \to B$ is a ring hom. Suppose $\psi(S) \subseteq B^{\times}$.

Then $\exists!\ \hat{\psi} : S^{-1}A \to B$ s.t.

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & \psi(a) \\ \frac{a}{1} \arrow[ur, mapsto] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & B \\ S^{-1}A \arrow[ur, dashed, "\hat{\psi}"'] & \end{tikzcd}$$

<u>Outline</u>. <u>Exist</u>. Let $\hat{\psi}(\frac{a}{s}) := \psi(s)^{-1}\psi(a)$

check that $\hat{\psi}$ is a ring hom.

<u>Uniq</u>. $\hat{\psi}(\frac{a}{1}) = \psi(a)\ \forall a \in A \Rightarrow \hat{\psi}(\frac{s}{1}) \cdot \hat{\psi}(\frac{1}{s}) = \hat{\psi}(\frac{s}{s})$

$$= \hat{\psi}(\tfrac{1}{1}) = \psi(1) = 1.$$

$\Rightarrow \hat{\psi}(\frac{1}{s}) = \psi(s)^{-1}$. And so

$\hat{\psi}(a/s) = \hat{\psi}(\frac{a}{1})\hat{\psi}(\frac{1}{s}) = \psi(a)\psi(s)^{-1}$. ■