# Homework 5

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1. Let $D$ be a PID; let $F$ be its field of fractions. Suppose $A \in M_{n,m}(D)$. Let $\underline{\ker(A)}(F) := \{v \in F^m \mid Av = 0\}$ and $\underline{\ker(A)}(D) := \{v \in D^m \mid Av = 0\}$, and $\underline{\mathrm{Im}(A)}(F) := \{Av \in F^n \mid v \in F^m\}$ and $\underline{\mathrm{Im}(A)}(D) := \{Av \in D^n \mid v \in D^m\}$.

(a) Prove that $D^m / \underline{\ker(A)}(D)$ is a free $D$-mod; and deduce $D^m$ has a $D$-basis $x_1, \dots, x_m$ s.t.

$$\underline{\ker(A)}(D) = Dx_{r+1} \oplus \cdots \oplus Dx_m,$$ where

$r = $ rank of $A$ as a matrix in $M_{n,m}(F)$.

(b) Let $r$ be as in part (a). Prove that $\exists$ a $D$-basis $y_1, \dots, y_n$ of $D^n$ and $d_1, \dots, d_r \in D$ s.t.

$$d_1 \mid d_2 \mid \cdots \mid d_r \text{ and } \underline{\mathrm{Im}(A)}(D) = D\, d_1 y_1 \oplus \cdots \oplus D\, d_r y_r.$$

(c) Let $x_i$'s be as in part (a). Prove that $\exists$ a $D$-basis $\{x_1', \dots, x_r'\}$ of $Dx_1 \oplus \cdots \oplus Dx_r$ s.t. $Ax_i' = d_i y_i$.

(d) Prove that $[x_1' \cdots x_r' \; x_{r+1} \cdots x_m] \in GL_m(D)$, $[y_1 \cdots y_n] \in GL_n(D)$, and

$$A[x_1' \cdots x_r' \; x_{r+1} \cdots x_m] = [y_1 \cdots y_n] \left[\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_r & \\ \hline & 0 & & 0 \end{array}\right].$$ Hence

$$A = \gamma_1 \left[\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_r & \\ \hline & 0 & & 0 \end{array}\right] \gamma_2,$$ where $\gamma_1 \in GL_n(D)$, $\gamma_2 \in GL_m(D)$, $d_1 \mid \cdots \mid d_r$.

(This is called a Smith form of $A$.)

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2. Let $A \in M_n(\mathbb{Z})$; and $M_A := \mathbb{Z}^n / \underline{\mathrm{Im}(A)}(\mathbb{Z})$.

(a) Show that $M_A$ is finite if and only if $\det(A) \neq 0$.

(b) Suppose $\det(A) \neq 0$. Prove that $|M_A| = |\det(A)|$.

( $\underline{\text{Hint}}$. Suppose $A = \gamma_1 \begin{bmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_m & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \gamma_2$ is a Smith form of $A$.

Show that $M_A \simeq \mathbb{Z}^{n-m} \oplus \bigoplus_{i=1}^{m} \mathbb{Z}/d_i\mathbb{Z}$ .)

3. Let $A \in M_n(k[x])$. Suppose $\det(A) \neq 0$. Prove that

$$\dim_k \left( k[x]^n / \underline{\mathrm{Im}(A)}(k[x]) \right) = \deg(\det(A)).$$

(Here $k$ is a field; and $\underline{\mathrm{Im}(A)}(k[x]) := \{ A v \mid v \in k[x]^n \}$.)

( $\underline{\text{Hint}}$. Suppose $A = \gamma_1 \begin{bmatrix} d_1(x) & & & & \\ & \ddots & & & \\ & & d_m(x) & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \gamma_2$ is a Smith form

of $A$. Show that $\underline{m=n}$ and

$$k[x]^n / \underline{\mathrm{Im}(A)}(k[x]) \simeq \bigoplus_{i=1}^{n} k[x]/\langle d_i(x) \rangle$$ .)

4. Let $k$ be a field, and $A \in M_n(k)$. Suppose

$$xI - A = \gamma_1 \begin{bmatrix} f_1(x) & & \\ & \ddots & \\ & & f_n(x) \end{bmatrix} \gamma_2$$

is a Smith form of $xI - A \in M_n(k[x])$; that means

$\gamma_1, \gamma_2 \in GL_n(k[x])$ and $f_1(x) \mid \cdots \mid f_n(x)$. Suppose $m$ is the

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largest integer such that $\deg f_{m-1} = 0$. Prove that

$$\begin{bmatrix} c(f_m) & & \\ & \ddots & \\ & & c(f_n) \end{bmatrix}$$ is the rational canonical form of $A$.

[Hint. ① $k[x]^n/(xI-A)k[x]^n \simeq k[x]/\langle f_1(x)\rangle \oplus \cdots \oplus k[x]/\langle f_n(x)\rangle$

(look at Hint of the previous problem).

$$= k[x]/\langle f_m(x)\rangle \oplus \cdots \oplus k[x]/\langle f_n(x)\rangle$$

as $k[x]$-modules.

② Argue why it is enough to show $k^n$ as $k[x]$-mod with action $x \cdot v := Av$ is isomorphic to $k[x]^n/(xI-A)\,k[x]^n$.

③ Let $\phi: k[x]^n \longrightarrow k^n$ be

$$\phi\Big(\sum_{i=0}^{m} x^i v_i\Big) := \sum_{i=0}^{m} A^i v_i \quad (\text{where } v_i \in k^n)$$

Show $\phi$ is a $k[x]$-mod. homomorphism; and

$$\ker \phi = (xI-A)\,k[x]^n.\ ]$$

5. Let $k$ be a field and $A \in M_n(k)$. Prove that $A$ is similar to its transpose $A^t$.

(Hint. Use rational canonical form and Problem 4.)

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6. In this problem, you will prove the following property for an integral domain, a unital commutative Noetherian ring, an arbitrary unital commutative ring. (The integral domain case is not needed to prove the general case; it is included because there is an easier argument in that case. So you do not need to write proof of part (a).)

(*) Suppose $M$ is an $A$-module and $d(M) = \operatorname{rank}(M) = n$ where $d(M)$ is the min. number of generators and $\operatorname{rank}(M)$ is the maximum number of linearly independent elements. Then $M \simeq A^n$.

(a) Prove (*) when $A$ is an integral domain.

(b) Prove (*) when $A$ is a Noetherian unital commutative ring.

(c) Prove (*) for an arbitrary unital commutative ring.

(Hint. For all the parts start with: $\exists\, \Phi, \psi$ s.t.

$$\begin{tikzcd} A^n \arrow[r, "\Phi", "\text{onto}"'] & M \\ & A^n \arrow[u, "\psi\ \text{injective}"'] \arrow[ul, dashed, "\psi'"] \end{tikzcd}$$

Using Hint of problem 7(c) from last week, $\exists\, \psi' : A^n \hookrightarrow A^n$

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Use this to deduce $\psi'(A^n) \oplus \ker(\phi) \subseteq A^n$ . (I)

In all the parts, you show $\underline{\ker(\phi)=0}$.

For <u>part (a)</u>, use (I) and rank $(A^n)$ to conclude $\ker \phi = 0$ .

For <u>part (b)</u>, use (I) and an argument similar to hint of problem 7(a) from last week.

For <u>part (c)</u>, suppose $(a_1, \ldots, a_n) \in \ker(\phi) \setminus \{0\}$ and $\psi'(e_i \cdot) = (a_{i1}, \ldots, a_{in})$ and use (I) and an argument similar to hint of problem 7(b) from last week.)

7. A matrix $N$ is called nilpotent if $\exists \ell \in \mathbb{Z}^+$, $N^\ell = 0$.

(a) Suppose $k$ is a field and $N \in M_m(k)$ is nilpotent. Prove that $N^m = 0$.

(b) Find two nilpotent matrices $N_1$ and $N_2$ that are NOT similar and have equal minimal polynomials.

(c) Prove that $N_1 \sim N_2$ if and only if, for any $j \in \mathbb{Z}^+$,

$$\dim \ker(N_1^j) = \dim \ker(N_2^j).$$

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8. Suppose $\{M_i\}_{i\in I}$ and $N$ are $R$-modules. Prove that

(a) $\mathrm{Hom}_R(\bigoplus_{i\in I} M_i, N) \simeq \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$

(b) $\mathrm{Hom}_R(N, \prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathrm{Hom}_R(N, M_i)$.

(as abelian groups).

9. Suppose $M$ is a simple $R$-mod. and let $D := \mathrm{End}_R(M)$.

(a) Prove that $\mathrm{End}_R(M^n) \simeq M_n(D)$ as rings.

(b) Suppose $M_i$'s are simple $R$-modules, and $M_i \not\simeq M_j$ as $R$-mod.

(b-1) For $\phi \in \mathrm{End}_R(\bigoplus_{i=1}^m M_i^{n_i})$, prove that

$$\phi(M_i^{n_i}) \subseteq M_i^{n_i}.$$

(b-2) Prove that $\mathrm{End}_R(\bigoplus_{i=1}^m M_i^{n_i}) \simeq M_{n_1}(D_1) \oplus \cdots \oplus M_{n_m}(D_m)$

as rings where $D_i = \mathrm{End}_R(M_i)$.

(c) Suppose $R \simeq M_1^{n_1} \oplus \cdots \oplus M_m^{n_m}$ as $R$-mod., where $M_i$'s are simple $R$-modules and $M_i \not\simeq M_j$. Prove that

$$R \simeq M_{n_1}(D_1^{op}) \oplus \cdots \oplus M_{n_m}(D_m^{op})$$

where $D_i = \mathrm{End}_R(M_i)$ are division rings.

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(Remark. Problem 9 is a part of Artin-Wedderburn's theorem.

Using Problem 8, in HW1 of math 200a, you can show that

$\mathbb{C}G \simeq M_1^{n_1} \oplus \cdots \oplus M_m^{n_m}$ as $\mathbb{C}G$-modules if $G$ is a finite gp.

And so by the above problem and showing $D_i = \mathbb{C}$, you get

$\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C})$; this gives us a lot of information

on irreducible characters of $G$. (it is the starting point of

representation theory of finite groups).)

10. (a) Let $\phi \in \operatorname{Hom}(\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z})$; let $e_j \in \prod_{i=1}^{\infty} \mathbb{Z}$ be $e_j(i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Suppose $\phi(e_j) = n_j \neq 0$ for any $j$.

Choose a sequence of positive integers $1 = k_1 < k_2 < \cdots$ s.t.

$\circledast$ $k_{j+1} \nmid k_j!\, n_j$. Consider $\Sigma := \{ (a_i)_{i=1}^{\infty} \mid a_i \in \{0, k_i!\}\}$. $\circledast$

(a-1) Argue why $\exists\, (a_i) \neq (a_i') \in \Sigma$ s.t. $\phi(a_i) = \phi(a_i')$.

(a-2) Suppose $i_0$ is the first index where $a_{i_0} \neq a_{i_0}'$.

Show $\phi((a_{i_0} - a_{i_0}') e_{i_0}) \notin k_{i_0+1}\mathbb{Z}$ (based on $\circledast$)

and $\phi((a_{i_0} - a_{i_0}') e_{i_0}) \in k_{i_0+1}\mathbb{Z}$ (based on $\circledast$)

And get a contradiction.

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(b) Use part (a) to deduce $\mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z}\right) \xrightarrow{\sim} \bigoplus_{i=1}^{\infty} \mathbb{Z}$,

$$\phi \longmapsto (\phi(e_i))$$

is an isomorphism.

(c) Use part (b) to show $\prod_{i=1}^{\infty} \mathbb{Z}$ is NOT a free abelian group.

(d) Use part (b) to show, $\mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z}\right) = 0$.

11. Suppose $R$ is a Noetherian ring and $\phi: R \to R$ is a surjective ring homomorphism. Prove that $\phi$ is an isomorphism.

(Hint: Consider $\ker(\phi^i)$.)

12. Suppose $A$ is a unital commutative ring.

(a) Suppose $\mathfrak{a} \trianglelefteq A$. Let $\sqrt{\mathfrak{a}} := \{a \in A \mid \exists n \in \mathbb{Z}^+, a^n \in \mathfrak{a}\}$.

Show that $\sqrt{\mathfrak{a}} \trianglelefteq A$ and $\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \mathfrak{a} \subseteq \mathfrak{p}}} \mathfrak{p}$.

(Hint: Show $\sqrt{\mathfrak{a}}/\mathfrak{a} = \mathrm{Nil}(A/\mathfrak{a})$ and look at lecture note 27, math 200a.)

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(b) Suppose $\langle f_1, \dots, f_n \rangle = A$ and $m_1, \dots, m_n \in \mathbb{Z}^+$.

Prove $\langle f_1^{m_1}, \dots, f_n^{m_n} \rangle = A$. (Hint. Use $\sqrt{\mathfrak{a}} \lhd A$.)

(c) Suppose $\langle f_1, \dots, f_n \rangle = A$. Let $M$ be an $A$-module.

Suppose $N \subseteq M$ is a submod, and $S_{f_i}^{-1} N = S_{f_i}^{-1} M$

for $1 \le i \le n$ where $S_{f_i} = \{1, f_i, f_i^2, \dots\}$.

Prove that $N = M$. (Hint. Let $x \in M$. Show $x \in N$ !)

(d) Suppose $\langle f_1, \dots, f_n \rangle = A$. Let $M$ be an $A$-module.

Suppose $S_{f_i}^{-1} M$ is a finitely generated $A_{f_i}$-mod.

Prove that $M$ is a finitely generated $A$-mod. (Hint Use (c))

(e) Suppose $\langle f_1, \dots, f_n \rangle = A$, and $A_{f_i}$'s are Noetherian.

Prove that $A$ is Noetherian. (Hint. Let $\mathfrak{a} \lhd A$, and use (d).)