Name: ______________________________

PID: ______________________________

Question	Points	Score
1	10	
2	5	
3	5	
4	10	
5	10	
Total:	40	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Prove that any element of finite order in $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable.

2. (5 points) Use the fact that $\mathbb{Z}[\sqrt{-10}]$ is not a UFD, and present a UFD D and a prime ideal $\mathfrak{p} \in \operatorname{Spec}(D)$ such that $D/\mathfrak{p}$ is not a UFD.

3. Suppose A is a proper unital subring of $\mathbb{Z}[i]$ which is not $\mathbb{Z}$.

   (a) (2 points) Prove that the field of fractions of A is $\mathbb{Q}[i]$. (Hint: think about it as a vector space over $\mathbb{Q}$)

   (b) (3 points) Prove that A is not a UFD.

4. (10 points) Let D be a PID and M be a finitely generated D-module. For any $\mathfrak{p} \in \operatorname{Max}(D)$, let $k(\mathfrak{p}) := D/\mathfrak{p}$. Notice that $M/\mathfrak{p}M$ is a $k(\mathfrak{p})$-vector space (You do not need to prove this). Let $d(M)$ be the minimum number of generators of M (as a D-module). Prove that

$$d(M) = \max_{\mathfrak{p} \in \operatorname{Max}(D)} \dim_{k(\mathfrak{p})} M/\mathfrak{p}M.$$

5. (a) (4 points) Let k be a field and $X \in M_n(k)$ be a nilpotent n-by-n matrix with entries in k; that means $X^m = 0$ for some positive integer m. Prove that $X^n = 0$.

(b) (1 point) Suppose A is an integral domain, and $X \in M_n(A)$ is nilpotent. Prove that $X^n = 0$.

(c) (5 points) Suppose A is a reduced ring; that means the nil-radical $\mathrm{Nil}(A) = 0$ of A is zero. Suppose $X \in M_n(A)$ is nilpotent. Prove that $X^n = 0$. (Hint: think about $A/\mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Spec}(A)$.

Good Luck!