

1. Lecture 7

Recall that if G is a group of order pq where $p < q$ are prime, then there exists a SES

$$1 \rightarrow C_q \rightarrow G \rightarrow C_p \rightarrow 1,$$

moreover, there exists an embedding $C_p \rightarrow G$. This leads us to the following definition.

DEFINITION 1. We say a SES

$$1 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 1$$

splits if there exists a group homomorphism $g : G_3 \rightarrow G_2$ such that

$$f_2 \circ g = \text{id}_{G_3}.$$

The group homomorphism g is called a *splitting homomorphism*. (We start from the margin go to the center and return.)

LEMMA 2. A standard SES $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$ splits if and only if there exists a subgroup H of G such that

- (1) (Transverse) $H \cap N = \{1\}$.
- (2) (Complement) $NH = G$.

For such a subgroup H , the following is an isomorphism of SESs.

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & G/N \longrightarrow 1 \\ & & \downarrow \text{id}_N & & \downarrow \text{id}_G & & \downarrow h \\ 1 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{h \circ \pi} & H \longrightarrow 1 \end{array}$$

where $G/N \xrightarrow{h} H \xrightarrow{i} G$ is a splitting map.

PROOF. ($\Rightarrow$) Suppose $h : G/N \rightarrow G$ is a splitting homomorphism. This means, for every $x \in G$,

$$(1) \quad \pi(h(xN)) = xN.$$

Let H denote the image of h .

Step 1. $H \cap N = \{1\}$.

PROOF OF STEP 1. Suppose $x \in H \cap N$. Since $x \in H$, it is in the image of h . So, $x = h(x'N)$ for some $x' \in G$. By (1), we deduce that

$$x'N = \pi(h(x'N)) = \pi(x) = xN.$$

Since $x \in N$, $xN = N$ is the neutral element of G/N . Therefore, $x = h(x'N) = h(N) = 1$. This complete the proof of Step 1. $\square$

Step 2. $NH = G$.

PROOF OF STEP 2. For $x \in G$, let $h := h(xN) \in H$. By (1), $hN = xN$. Therefore, $x \in Nh$, which implies that $x = nh$ for some $n \in N$. This completes the proof. $\square$

($\Leftarrow$) We begin by recalling the second isomorphism theorem

THEOREM 3 (The second isomorphism theorem.). *Suppose N is a normal subgroup of G and H is a subgroup. Then*

$$\frac{H}{N \cap H} \rightarrow \frac{NH}{N}, \quad h(N \cap H) \mapsto hN$$

is an isomorphism.

PROOF. Let $f : H \rightarrow G/N$, $f(h) := hN$. Then,

$$\text{Im } f = NH/N \quad \text{and} \quad \ker f = H \cap N.$$

Therefore, by the first isomorphism theorem, the claim follows. $\square$

By the second isomorphism theorem and the given conditions on N and H , we deduce that

$$H \rightarrow G/N, \quad h \mapsto hN$$

is a group isomorphism. Let $h : G/N \rightarrow H$ be the inverse of the above isomorphism.

Claim. h is a splitting homomorphism.

PROOF OF CLAIM. We must show that for every

$$xN \in G/N, \quad \pi(h(xN)) = xN.$$

Let $h := h(xN)$. By the definition of h , we have $hN = xN$, which means $\pi(h) = xN$. This completes the proof. $\square$

The last claim that $(\text{id}_N, \text{id}_G, h)$ is a SES isomorphism is clear. $\square$

PROPOSITION 4. *The SES $1 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 1$ splits if and only if there exists a subgroup H of G_2 such that*

$$\ker f_2 \cap H = \{1\} \quad \text{and} \quad G_2 = (\ker f_2)H.$$

For this given subgroup H , the following is an isomorphism of SESs for some group homomorphism $h : G_2 \rightarrow H$,

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\ & & \bar{f}_1^{-1} \uparrow & & \text{id}_{G_2} \uparrow & & \uparrow f_2|_H \\ 1 & \longrightarrow & \ker f_2 & \xrightarrow{\iota} & G_2 & \xrightarrow{h} & H \longrightarrow 1 \end{array}$$

We begin with proving the following Lemma.

LEMMA 5. *If two SESs are isomorphic, one of them splits if and only if the other one does.*

PROOF. Suppose the following is a commuting diagram where the rows are SESs, and the vertical arrows are isomorphisms, and h is a splitting group homomorphism.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xleftarrow[h]{f_2} & G_3 & \longrightarrow & 1 \\ & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 & & \\ 1 & \longrightarrow & H_1 & \xrightarrow{g_1} & H_2 & \xrightarrow{g_2} & H_3 & \longrightarrow & 1 \end{array}$$

Then, define $\bar{h} : H_3 \rightarrow H_2$, $\bar{h} := \theta_2 \circ h \circ \theta_3^{-1}$. We show that $\bar{h}$ is a splitting group homomorphism of the second SES.

$$\begin{aligned} g_2 \circ \bar{h} &= g_2 \circ (\theta_2 \circ h \circ \theta_3^{-1}) \\ &= (g_2 \circ \theta_2) \circ h \circ \theta_3^{-1} \\ &= \theta_3 \circ \underbrace{f_2 \circ h \circ \theta_3^{-1}}_{\text{id}_{G_3}} \\ &= \text{id}_{H_3}. \end{aligned}$$

Therefore, if the first SES splits, then the second one splits. This completes the proof. $\square$

PROOF OF PROPOSITION 4. Recall that $(\hat{f}_1^{-1}, \text{id}_{G_2}, \bar{f}_2)$ is an isomorphism of SESs

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \longrightarrow & 1 \\ & & \tilde{f}_1^{-1} \uparrow & & \text{id} \uparrow & & \bar{f}_2 \uparrow & & \\ 1 & \longrightarrow & \ker f_2 & \longrightarrow & G_2 & \longrightarrow & G_2 / \ker f_2 & \longrightarrow & 1 \end{array}$$

where these maps are defined in Lemma ???. Hence, by Lemma 5, the first row splits if and only if the second row does. The second row is a standard SES, and so by Lemma 2, the second row splits if and only if there exists a subgroup H of G_2 such that

$$\ker f_2 \cap H = \{1\} \quad \text{and} \quad G_2 = (\ker f_2)H.$$

Moreover, by Lemma 2, we obtain the following SES isomorphisms.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \longrightarrow & 1 \\ & & \hat{f}_1^{-1} \uparrow & & \text{id}_{G_2} \uparrow & & \bar{f}_2 \uparrow & & \\ 1 & \longrightarrow & \ker f_2 & \xrightarrow{\iota} & G_2 & \xrightarrow{\pi} & G_2 / \ker f_2 & \longrightarrow & 1 \\ & & \downarrow \text{id}_{\ker f_2} & & \downarrow \text{id}_{G_2} & & \downarrow h & & \\ 1 & \longrightarrow & \ker f_2 & \xrightarrow{\iota} & G_2 & \xrightarrow{h \circ \pi} & H & \longrightarrow & 1 \end{array}$$

The last claim follows from the observation

$$\bar{f}_2(h^{-1}(x)) = \bar{f}_2(x \ker f_2) = f_2(x).$$

This completes the proof. $\square$

Given groups G_1 and G_3 , if the SES $1 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 1$ splits, can we describe the middle group G_2 ?

By Lemma 2, every element of G can be written as a product of an element of H and an element of N . So, we may use H and N as a *coordinate system* to capture elements of G . Now, the question is how we can describe multiplication of two elements. Note

$$(n_1 h_1)(n_2 h_2) = (n_1 \underbrace{(h_1 n_2 h_1^{-1})}_{\in N})(h_1 h_2).$$

Hence, if we know how elements of H act on N by conjugation, then we can multiply any two elements of G . This leads us to the definition of semidirect product of two groups.

LEMMA 6. Suppose N and H are groups and $f : H \rightarrow \text{Aut}(N)$ is a group homomorphism. For $(n_1, h_1), (n_2, h_2) \in N \times H$, we define

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1(f(h_1))(n_2), h_1 h_2).$$

Then $(N \times H, \cdot)$ forms a group.

PROOF. We leave it as an exercise. $\square$

The group introduced in Lemma 6 is called a *semidirect product* of N and H , and it is denoted by $N_f \rtimes H$, or simply $N \rtimes H$.

Recall that having a group homomorphism $f : H \rightarrow \text{Aut}(N)$ induces an action of H on N . We sometimes write ${}^h n$ to denote $(f(h))(n)$.

PROPOSITION 7. To every semidirect product $N \rtimes H$, we can associate a canonical splitting SES as follows:

$$1 \rightarrow N \xrightarrow{j_1} N \rtimes H \xrightarrow{p_2} H \rightarrow 1$$

with a splitting homomorphism

$$j_2 : H \rightarrow N \rtimes H, j_2(h) := (1, h),$$

where $j_1(n) := (n, 1)$ and $p_2(n, h) = h$.

PROOF. For every $n_1, n_2 \in N$,

$$j_1(n_1)j_1(n_2) = (n_1, 1) \cdot (n_2, 1) = (n_1 f(1)(n_2), 1) = (n_1 n_2, 1) = j_1(n_1 n_2).$$

Hence, j_1 is a group homomorphism.

For every $(n_1, h_1), (n_2, h_2) \in N \rtimes H$, we have

$$p_2((n_1, h_1) \cdot (n_2, h_2)) = p_2(n_1 {}^{h_1} n_2, h_1 h_2) = h_1 h_2 = p_2(n_1, h_1) p_2(n_2, h_2).$$

Thus, p_2 is a group homomorphism.

For every $h_1, h_2 \in H$,

$$j_2(h_1)j_2(h_2) = (1, h_1) \cdot (1, h_2) = (1f(h_1)(1), h_1h_2) = (1, h_1h_2) = j(h_1h_2).$$

The maps j_1 and j_2 are clearly injective, and the map p_2 is clearly surjective. Moreover,

$$p_2(j_2(h)) = p_2(1, h) = h,$$

and so j_2 is a splitting homomorphism. $\square$

It is often more illuminating to work with the copies $j_1(N)$ and $j_2(H)$ of N and H , respectively. The key relation in the semidirect product $N_f \rtimes H$ is given by

$$j_2(h)j_1(n)j_2(h)^{-1} = {}^h n$$

where the right hand side is $(f(h))(n)$. We sometimes drop the *decorations* j_1 and j_2 , and simply regard N and H as subgroups of the semidirect product.

The next result shows that every splitting SES is isomorphic to a canonical SES of a semidirect product.

PROPOSITION 8. *Suppose $1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 1$ is a splitting SES. Then*

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\ & & \text{id}_{G_1} \uparrow & & m \uparrow & & \text{id}_{G_3} \uparrow \\ 1 & \longrightarrow & G_1 & \xrightarrow{j_1} & G_1 \rtimes G_3 & \xrightarrow{p_2} & G_3 \longrightarrow 1 \end{array}$$

gives a SES isomorphism, where

$$f : G_3 \rightarrow \text{Aut}(G_1), \quad (f(x_3))(x_1) := x_3 x_1 x_3^{-1},$$

$$m : G_1 \rtimes G_3, \quad m(x_1, x_3) := f_1(x_1)h(x_3),$$

and $h : G_3 \rightarrow G_2$ is a splitting map. In particular,

$$G_2 \simeq G_1 \rtimes G_3.$$

We will prove this Proposition in the next lecture.