

1. Lecture 6

(First we completed the proof of Proposition ??.) Let us recall that, by Lemma ??, if G is a group of order pq where $p < q$ are primes, then G has a characteristic subgroup Q of order q . We also know that G has a Sylow p -subgroup P . Hence,

$$P \simeq C_p, \quad Q \simeq C_q, \quad \text{and} \quad G/Q \simeq C_p$$

where C_n denotes a cyclic group of order n . This leads us to an extremely important question:

Given N and G/N , can we describe the structure of G ? To what extent is this possible?

DEFINITION 1. (1) A *co-chain* (in the category of groups) is given by a sequence $\{G_i\}_i$ of groups and a sequence $f_i : G_i \rightarrow G_{i+1}$ of group homomorphisms:

$$\cdots \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots$$

The index set of a co-chain might be finite.

(2) A *co-chain complex* is a co-chain $(G_\bullet, f_\bullet)$ such that, for every i , $f_{i+1} \circ f_i$ is trivial; equivalently,

$$\text{Im}(f_i) \subseteq \ker f_{i+1}.$$

(3) An *exact sequence* is a co-chain complex $(G_\bullet, f_\bullet)$ such that

$$\text{Im}(f_i) = \ker f_{i+1}.$$

(4) A *short exact sequence* (SES) is an exact sequence of the form

$$1 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 1.$$

(5) A homomorphism between two co-chains $(G_\bullet, f_\bullet)$ and $(H_\bullet, g_\bullet)$ is a sequence of group homomorphisms $\theta_i : G_i \rightarrow H_i$ such that

$$\begin{array}{ccccccc} \cdots & \xrightarrow{f_{i-1}} & G_i & \xrightarrow{f_i} & G_{i+1} & \xrightarrow{f_{i+1}} & \cdots \\ & & \downarrow \theta_i & & \downarrow \theta_{i+1} & & \\ \cdots & \xrightarrow{g_{i-1}} & H_i & \xrightarrow{g_i} & H_{i+1} & \xrightarrow{g_{i+1}} & \cdots \end{array}$$

is a commuting diagram; this means

$$g_i \circ \theta_i = \theta_{i+1} \circ f_i$$

for all i . We say $\{\theta_i\}$ is an isomorphism of the co-chains if θ_i is an isomorphism for every i .

(6) We say two SESs are isomorphic if they are isomorphic as co-chains.

EXAMPLE 2. (Standard SES) If $N \trianglelefteq G$, then

$$1 \rightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \rightarrow 1$$

is a SES where $\iota(x) = x$ and $\pi(x) := xN$. This type of SES is called a *standard SES*.

LEMMA 3. (1) The co-chain $1 \rightarrow G_1 \xrightarrow{f_1} G_2$ is exact if and only if f_1 is injective.

(2) The co-chain $G_1 \xrightarrow{f_1} G_2 \rightarrow 1$ is exact if and only if f_1 is surjective.

PROOF. (1) The given co-chain is exact if and only if $\{1\} = \ker f_1$. We have that $\ker f_1 = \{1\}$ precisely when f_1 is injective.

(2) The given co-chain is exact if and only if $\text{Im}(f_1)$ is the kernel of the trivial map $G_2 \rightarrow 1$. This holds precisely when f_1 is surjective. $\square$

LEMMA 4. Suppose

$$1 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_2 \rightarrow 1$$

is a SES. Then there exist isomorphisms

$$\tilde{f}_1 : G_1 \rightarrow \ker f_2 \quad \text{and} \quad \bar{f}_2 : G_2 / \ker f_2 \rightarrow G_2$$

such that

$$(1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_2 \longrightarrow 1 \\ & & \tilde{f}_1^{-1} \uparrow & & \text{id} \uparrow & & \bar{f}_2 \uparrow \\ 1 & \longrightarrow & \ker f_2 & \longrightarrow & G_2 & \longrightarrow & G_2 / \ker f_2 \longrightarrow 1 \end{array}$$

is a commuting diagram. In particular, $(\tilde{f}_1^{-1}, \text{id}_{G_2}, \bar{f}_2)$ is an isomorphism of SESs, and every SES is isomorphic to a standard SES.

PROOF. By the first isomorphism theorem,

$$\bar{f}_2 : G_2 / \ker f_2 \rightarrow \text{Im}(f_2), \quad \bar{f}_2(x \ker f_2) := f_2(x)$$

is a well-defined isomorphism. Since $\{(G_i, f_i)\}$ is a SES, $\text{Im} f_1 = \ker f_2$ and f_1 is injective. Therefore,

$$\tilde{f}_1 : G_1 \rightarrow \ker f_2, \quad \tilde{f}_1(x) := f_1(x)$$

is an isomorphism. It is remained to check why (1) is a commuting diagram.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\
 & & \uparrow \tilde{f}_1^{-1} & & \uparrow \text{id} & & \uparrow \bar{f}_2 \\
 & & \tilde{f}_1^{-1}(x) \longmapsto x & & x \longmapsto f_2(x) & & \\
 & & \uparrow & & \uparrow & & \\
 & & x \longmapsto x & & x \longmapsto x \ker f_2 & & \\
 1 & \longrightarrow & \ker f_2 & \xrightarrow{\iota} & G_2 & \xrightarrow{\pi} & G_2 / \ker f_2 \longrightarrow 1
 \end{array}$$

□

Given information on the *margins*, what can we say for the middle part which is presumably the more complicated section?

LEMMA 5 (Short five lemma). *Suppose $(\theta_1, \theta_2, \theta_3)$ is a homomorphism of SESs. Then the following hold.*

- (1) *If θ_1, θ_3 are injective, then θ_2 is injective.*
- (2) *If θ_1, θ_3 are surjective, then θ_2 is surjective.*
- (3) *If θ_1, θ_3 are isomorphisms, then θ_2 is an isomorphism.*

PROOF. We follow the arrows to prove the claims.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\
 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 \\
 & & x_1 \textcircled{2} \longmapsto x_2 \textcircled{2} & & x_2 \textcircled{1} \longmapsto f_2(x_2) \textcircled{1} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \theta_1(x_1) \textcircled{2} \longmapsto g_1(\theta_1(x_1)) \textcircled{2} & & 1 \textcircled{1} \longmapsto 1 \textcircled{1} & & \\
 1 & \longrightarrow & H_1 & \xrightarrow{g_1} & H_2 & \xrightarrow{g_2} & H_3 \longrightarrow 1
 \end{array}$$

To show θ_2 is injective, we show that its kernel is trivial. Suppose x_2 is in the kernel of θ_2 . Then following the arrows, we obtain the square whose vertices are marked by ①. In particular, it follows that

$$x_2 \in \ker f_2 = \text{Im } f_1.$$

This implies that $x_2 = f_1(x_1)$ for some $x_1 \in G_1$. Starting with x_1 , we write the square whose vertices are marked by ②. It follows that $g_1(\theta_1(x_1)) = 1$. Since g_1 and θ_1 are injective, we deduce that $x_1 = 1$. Thus $x_2 = f_1(x_1) = 1$. This completes the proof of part (1).

To prove Part (2), we must show every $y_2 \in H_2$ is in the image of θ_2 .

$$\begin{array}{ccccccc}
 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\
 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 \\
 1 & \longrightarrow & H_1 & \xrightarrow{g_1} & H_2 & \xrightarrow{g_2} & H_3 \longrightarrow 1
 \end{array}$$

$x_2 \textcircled{3} \xrightarrow{\text{green}} f_2(x_2) \textcircled{3} = x_3 \textcircled{2} \xrightarrow{\text{red}} \textcircled{3} = g_2(y_2) \textcircled{1}$
 $\theta_2(x_2) \textcircled{3}, y_2 \textcircled{1} \xrightarrow{\text{blue}} \textcircled{3} = g_2(y_2) \textcircled{1}$

Since θ_3 is surjective, there exists $x_3 \in G_3$ such that $\theta_3(x_3) = y_3$. Since f_3 is surjective, there exists $x_2 \in G_2$ such that $f_2(x_2) = x_3$. Hence, we get the square labeled by $\textcircled{3}$. In particular,

$$g_2(\theta_2(x_2)) = g_2(y_2).$$

Therefore, $\theta_2(x_2)^{-1}y_2 \in \ker g_2 = \text{Im } g_1$. Thus, $\theta_2(x_2)^{-1}y_2 = g_1(y_1)$ for some $y_1 \in H_1$. Since θ_1 is surjective, there exists $x_1 \in G_1$ such that $\theta_1(x_1) = y_1$. We obtain the square labeled $\textcircled{4}$.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 \longrightarrow 1 \\
 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 \\
 1 & \longrightarrow & H_1 & \xrightarrow{g_1} & H_2 & \xrightarrow{g_2} & H_3 \longrightarrow 1
 \end{array}$$

$x_1 \textcircled{4} \xrightarrow{\text{red}} f_1(x_1) \textcircled{4} \xrightarrow{\text{red}} \theta_2(x_2)^{-1}y_2 \textcircled{4}$
 $y_1 \textcircled{4} \xrightarrow{\text{red}} \theta_2(x_2)^{-1}y_2 \textcircled{4}$

Therefore, $\theta_2(x_2)^{-1}y_2 \in \text{Im } \theta_2$. It implies that $y_2 \in \text{Im } \theta_2$. This completes the proof of Part (2).

If θ_1 and θ_3 are isomorphisms, then by Parts (1) and (2), θ_2 is both injective and surjective. Hence, θ_2 is an isomorphism. This completes the proof. $\square$

Going back to groups of order pq , recall that every such group fits into a SES

$$1 \rightarrow C_q \rightarrow G \rightarrow C_p \rightarrow 1,$$

and that C_p embeds into G . This motivates the definition of a split short exact sequence, which we will discuss in the next lecture.