

1. Lecture 3

Recall that, assuming G acts on a set X , we say $x, x' \in X$ are G -similar if there exists $g \in G$ such that $x' = g \cdot x$. The set of all points that are G -similar to x is called the G -orbit of x and it is denoted by $G \cdot x$. So,

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

The G -similarity defines a relation on the set X .

LEMMA 1. *Suppose $G \curvearrowright X$.*

- (1) *The G -similarity relation is an equivalence relation.*
- (2) *The set $\{G \cdot x \mid x \in X\}$ of G -orbits is a partition of X .*

PROOF. (1) We write $x \sim x'$ if $x' = g \cdot x$ for some $g \in G$.

Reflexive. For every $x \in X$, $x = e_G \cdot x$. Hence $x \sim x$.

Symmetric. If $x \sim x'$, then $x' = g \cdot x$ for some $g \in G$. Hence,

$$g^{-1} \cdot x' = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e_G \cdot x = x.$$

Therefore, $x' \sim x$.

Transitive. If $x \sim x'$ and $x' \sim x''$, then there exist $g, g' \in G$ such that

$$x' = g \cdot x \quad \text{and} \quad x'' = g' \cdot x'.$$

Hence

$$x'' = g' \cdot (g \cdot x) = (g'g) \cdot x,$$

which implies that $x \sim x''$.

(2) Note that G -orbits are equivalence classes of the equivalence relation $\sim$.

Hence, they form a partition of X . $\square$

Let us recall why equivalence classes form a partition. Given an equivalence relation $\sim$ on a set X , the equivalence class $[x]$ of $x \in X$ is

$$[x] := \{x' \in X \mid x \sim x'\}.$$

LEMMA 2. *In the above setting, for $x, x' \in X$, the following statements are equivalent.*

- (1) $x' \in [x]$.
- (2) $[x'] \subseteq [x]$.
- (3) $[x'] = [x]$.

(4) $[x] \cap [x'] \neq \emptyset$.

PROOF. (1) $\Rightarrow$ (2).

$$\left. \begin{array}{l} \forall y \in [x'], \quad x' \sim y, \\ x' \in [x] \text{ implies } x \sim x' \end{array} \right\} \implies x \sim y \text{ which implies } y \in [x].$$

(2) $\Rightarrow$ (3). Since $x' \sim x'$, $x' \in [x']$ which implies $x' \in [x]$. Therefore, $x \sim x'$, and so $x' \sim x$. Thus $x \in [x']$. Using the implication (1) $\Rightarrow$ (2), we deduce $[x] \subseteq [x']$. Hence, $[x] = [x']$.

(3) $\Rightarrow$ (4) is clear as $x \in [x]$.

(4) $\Rightarrow$ (1). Suppose $y \in [x] \cap [x']$. Using the implication (1) $\Rightarrow$ (3), we deduce that $[y] = [x]$ and $[y] = [x']$. Therefore, $[x] = [x']$, which implies that $x' \in [x]$. This completes the proof. $\square$

COROLLARY 3. *Suppose $\sim$ is an equivalence relation on X . Then the set $\{[x] \mid x \in X\}$ is a partition of X .*

PROOF. By Lemma 2, two distinct equivalence classes are disjoint. We also note that $x \in [x]$ for all $x \in X$. This completes the proof. $\square$

By Lemma 1 and Lemma 2, we deduce the following corollary.

COROLLARY 4. *Suppose $G \curvearrowright X$. Then for every $x, x' \in X$, the following statements are equivalent.*

- (1) $x' \in G \cdot x$.
- (2) $G \cdot x = G \cdot x'$.
- (3) $G \cdot x \cap G \cdot x' \neq \emptyset$.

Given a group action $G \curvearrowright X$, the partition of all the G -orbits is denoted by $G \backslash X$, and it is called the quotient of X by G .

PROPOSITION 5. *Suppose X is a finite set and $G \curvearrowright X$. Then*

$$|X| = \sum_{\mathcal{O} \in G \backslash X} |\mathcal{O}|.$$

The following theorem is fundamental to the study of G -orbits

THEOREM 6 (The orbit-stabilizer theorem). *Suppose $G \curvearrowright X$. Then for every $x \in X$ the following statements are satisfied.*

(1) Let $G_x := \{g \in G \mid g \cdot x = x\}$. Then G_x is a subgroup of G , and it is called the stabilizer of x .

(2) The following is a well-defined bijection:

$$G/G_x \rightarrow G \cdot x, \quad gG_x \mapsto g \cdot x.$$

(3) $|G \cdot x| = [G : G_x]$.

PROOF. (1) Suppose $g, g' \in G_x$. Then

$$(gg') \cdot x = g \cdot \underbrace{(g' \cdot x)}_{\text{using } g' \in G_x} = \underbrace{g \cdot x}_{\text{using } g \in G_x} = x.$$

Hence, $gg' \in G_x$.

If $g \in G_x$, then $g \cdot x = x$, which implies that $g^{-1} \cdot x = x$. So $g^{-1} \in G_x$.

Finally, $e_G \cdot x = x$ implies that $e_G \in G_x$.

(2) **Well-defined and injective** Note

$$\begin{aligned} gG_x = g'G_x &\iff g^{-1}g' \in G_x \\ &\iff (g^{-1}g') \cdot x = x \\ &\iff g' \cdot x = g \cdot x. \end{aligned}$$

Surjectivity is clear.

(3) This is an immediate consequence of Part (2). □

COROLLARY 7. Suppose G and X are finite and $G \curvearrowright X$. Then

$$(1) \quad |X| = \sum_{G \cdot x \in G \backslash X} \frac{|G|}{|G_x|}.$$

Given a group action $G \curvearrowright X$, we say $x \in X$ is a fixed point of G if for all $g \in G$, $g \cdot x = x$. So, x is a fixed point of G if and only if $|G \cdot x| = 1$ if and only if $G_x = G$. The set of fixed points of G is denoted by either X^G or $\text{Fix}(G)$. Thus,

(1) can be rewritten as

$$(2) \quad |X| = |X^G| + \sum_{G \cdot x \in G \backslash X, |G \cdot x| > 1} \frac{|G|}{|G_x|}.$$

We refer to (2) as the *generalized class equation*.

Let us see consequences of the above results for the conjugation action of G on itself.

PROPOSITION 8. Consider the conjugation action $G \curvearrowright G$. Then the following statements hold.

- (1) For every $x \in G$, the stabilizer subgroup of x is the centralizer $C_G(x)$ of x in G .
- (2) The G -orbit of x is the conjugacy class of x in G :

$$\text{cl}(x) := \{gxg^{-1} \mid g \in G\}.$$

- (3) $|\text{cl}(x)| = [G : C_G(x)]$.
- (4) Conjugacy classes form a partition of G .
- (5) (Class equation) $|G| = |Z(G)| + \sum_{|\text{cl}(x)| > 1} [G : C_G(x)]$.

PROOF. Note that $g \in G_x$ if and only if

$$g \cdot x = x \iff gxg^{-1} = x \iff g \in C_G(x).$$

We also deduce that the set of G -fixed points is the center $Z(G)$ of G .

The rest are immediate consequences of the orbit-stabilizer theorem and (2). □

Another interesting action is the action of G on the set $\text{Sub}(G)$ of its subgroups by conjugation.

PROPOSITION 9. Let $\text{Sub}(G)$ be the set of all subgroups of G . For $g \in G$ and $H \in \text{Sub}(G)$, define

$$g \cdot H := gHg^{-1}.$$

Then the following statements hold.

- (1) The above $\cdot$ defines a group action.
- (2) The G -orbit of H consists of conjugates of H .
- (3) The stabilizer subgroup of H is

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}.$$

It is called the normalizer of H in G .

- (4) $H \subseteq N_G(H)$.
- (5) N is fixed by G if and only if it is a normal subgroup.
- (6) The number of conjugates of H is equal to $[G : N_G(H)]$.

PROOF. This is an immediate consequence of the orbit-stabilizer theorem. □

The importance of whether a point $x \in X$ is fixed by an element $g \in G$ can be seen in the role of the stabilizer subgroup in the orbit-stabilizer theorem. We may also fix the parameter g in the equation $g \cdot x = x$, and consider the set of fixed points of g . The set of g -fixed points is denoted by X^g . We are obtaining two functions:

$$(3) \quad X \rightarrow \text{Sub}(G), x \mapsto G_x \quad \text{and} \quad G \rightarrow P(X), g \mapsto X^g.$$

Recall that $G \curvearrowright \text{Sub}(G)$ by conjugation, and note that $G \curvearrowright P(X)$ by

$$g \cdot Y := \{g \cdot y \mid y \in Y\}.$$

To understand the next result from the proper perspective, we define the category of G -sets. An object in the category of G -sets is a set Z together with a G -action $a : G \times Z \rightarrow Z$. A morphism f between two G -sets (Z_1, a_1) and (Z_2, a_2) is a function $f : Z_1 \rightarrow Z_2$ such that, for all $g \in G$ and $z_1 \in Z_1$,

$$f(a_1(g, z_1)) = a_2(g, f(z_1)).$$

We often write this condition as

$$f(g \cdot z_1) = g \cdot f(z_1).$$

A G -set map is also called a G -equivariant map.

LEMMA 10. *The functions $X \rightarrow \text{Sub}(G)$, $x \mapsto G_x$ and $G \mapsto P(X)$, $g \mapsto X^g$ are G -equivariant, where the G -action on the G -sides are via conjugation and on the X -side via the given action.*

Note that the first map is G -equivariant if and only if

$$G_{g \cdot x} = gG_xg^{-1}$$

for all $x \in X$ and $g \in G$. The second map is G -equivariant if and only if

$$X^{g'gg'^{-1}} = g' \cdot X^g$$

for all $g, g' \in G$.

PROOF.

$$\begin{aligned} g' \in G_{g \cdot x} &\iff g' \cdot (g \cdot x) = g \cdot x \iff (g^{-1}g'g) \cdot x = x \\ &\iff g^{-1}g'g \in G_x \iff g' \in gG_xg^{-1}. \end{aligned}$$

This shows that the first map is G -equivariant.

$$\begin{aligned}
x \in X^{g'gg'^{-1}} &\iff (g'gg'^{-1}) \cdot x = x \iff g \cdot (g'^{-1} \cdot x) = g'^{-1} \cdot x \\
&\iff g'^{-1} \cdot x \in X^g \iff x \in g' \cdot X^g.
\end{aligned}$$

This completes the proof. $\square$

In any classification problem, we are interested in describing objects up to isomorphism. We often begin with a larger set of parameters and then identify two collections of parameters whenever the corresponding objects are isomorphic. The set of isomorphism classes can often be identified with the set of G -orbits for a suitable group G . Hence, understanding the quotient set $G \backslash X$ is very important.

The number of elements of the quotient space also has applications in enumerative combinatorics. For instance, consider the problem of counting necklaces with n black beads and m white beads. One starts with the set $X_{n,m}$ of labeled beads, and then lets the dihedral group $D_{2(m+n)}$ act on $X_{n,m}$. The desired number is the cardinality of the quotient space $D_{2(m+n)} \backslash X_{n,m}$.

THEOREM 11 (Cauchy-Frobenius theorem). *Suppose G and X are finite and $G \curvearrowright X$. Then*

$$|G \backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Note that in plain language, the Cauchy-Frobenius theorem states that the number of G -orbits is the average of the number of fixed points of elements of G .

This theorem is published in a book by Burnside, so it is sometimes called Burnside's lemma.

PROOF. Let $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$. We count the number of elements of F in two different ways. First fixing the G -component:

$$(4) \quad |F| = \sum_{g \in G} |F \cap \{g\} \times X| = \sum_{g \in G} |X^g|.$$

Second fixing the X -component:

$$\begin{aligned}
(5) \quad |F| &= \sum_{x \in X} |F \cap G \times \{x\}| = \sum_{x \in X} |G_x| \\
&= \sum_{\mathcal{O} \in G \backslash X} \sum_{x \in \mathcal{O}} |G_x| \quad (\text{as } G \backslash X \text{ is a partition of } X).
\end{aligned}$$

Note that for every orbit $\mathcal{O}$ and $x \in \mathcal{O}$, $\mathcal{O} = G \cdot x$. By the orbit-stabilizer theorem,

$$|\mathcal{O}| = |G \cdot x| = \frac{|G|}{|G_x|},$$

which implies that, for all $x \in \mathcal{O}$,

$$(6) \quad |G_x| = \frac{|G|}{|\mathcal{O}|}.$$

By (5) and (6), we deduce

$$(7) \quad |F| = \sum_{\mathcal{O} \in G \backslash X} \sum_{x \in \mathcal{O}} \frac{|G|}{|\mathcal{O}|} = |G| \sum_{\mathcal{O} \in G \backslash X} 1 = |G| |G \backslash X|.$$

By (4) and (7), we deduce

$$|G| |G \backslash X| = \sum_{g \in G} |X^g|,$$

which completes the proof. □