

1. Lecture 2.

For a given group G and an object X , can G be viewed as “symmetries” of X ? This means what can we say about group homomorphisms $G \rightarrow \text{Symm}(X)$? Given a group homomorphism $\theta : G \rightarrow \text{Symm}(X)$, we may use elements of G as symmetries of X and *move* them:

$$g \cdot x := \theta(g)(x).$$

Under the above definition, $1 \cdot x = \theta(1)(x) = x$ for all $x \in X$, and for every $g_1, g_2 \in G$,

$$g_1 \cdot (g_2 \cdot x) = \theta(g_1)(\theta(g_2)(x)) = \theta(g_1 g_2)(x) = (g_1 g_2) \cdot x.$$

This leads us to the definition of *group action*.

DEFINITION 1. Suppose G is a group and X is a set. A group action of G on X is a function

$$a : G \times X \rightarrow X, \quad a(g, x) := g \cdot x$$

satisfying the following properties:

$$\forall x \in X, 1 \cdot x = x \quad \text{and} \quad \forall g_1, g_2 \in G, \forall x \in X, g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x.$$

We often write $G \curvearrowright X$ and read this as G acts on X .

When X is an object of a category $\mathcal{C}$ and, for every $g \in G$, the map $x \mapsto g \cdot x$ is an arrow of $\mathcal{C}$, we say G acts via $\mathcal{C}$ -automorphisms. For instance, the following are interesting examples: where X is a metric space and G acts on X by isometries, and where X is a group and G acts on X by automorphisms.

By the above discussion, we have a general method of constructing group actions:

Given a group homomorphism $\theta : G \rightarrow S_X$,

$$a_\theta(g, x) := \theta(g)(x)$$

defines an action of G on X , and it is called the action induced by θ .

In fact, every action is of this form.

PROPOSITION 2. Let $\mathcal{A}(G, X) := \{a : G \times X \rightarrow X \mid a \text{ is an action}\}$. Define

$$\Phi : \text{Hom}(G, S_X) \rightarrow \mathcal{A}(G, X), \quad \theta \mapsto a_\theta,$$

and

$$\Psi : \mathcal{A}(G, X) \rightarrow \text{Hom}(G, S_X), \quad a \mapsto \theta_a,$$

where

$$\theta_a(g)(x) := a(g, x).$$

Then Φ and Ψ are well-defined and inverses of each other.

PROOF. We have already proved that a_θ is an action. Hence, Φ is well-defined.

We prove that Ψ is well-defined in several steps.

Step 1. For every $g_1, g_2 \in G$, $\theta_a(g_1) \circ \theta_a(g_2) = \theta_a(g_1 g_2)$.

For every $x \in X$,

$$\begin{aligned} (\theta_a(g_1) \circ \theta_a(g_2))(x) &= \theta_a(g_1)(g_2 \cdot x) = g_1 \cdot (g_2 \cdot x) \\ &= (g_1 g_2) \cdot x = \theta_a(g_1 g_2)(x). \end{aligned}$$

This completes the proof of Step 1.

Step 2. For every $g \in G$, $\theta_a(g) \in S_X$.

By Step 1, $\theta_a(g) \circ \theta_a(g^{-1}) = \theta_a(g^{-1}) \circ \theta_a(g) = \theta_a(1) = 1_X$. This implies that $\theta_a(g)$ is invertible. This completes the proof of Step 2.

By Steps 1 and 2, we deduce that $\theta_a : G \rightarrow S_X$ is a group homomorphism. Hence, Ψ is well-defined.

Step 3. $\Psi(\Phi(a)) = a$

For $g \in G$ and $x \in X$,

$$\Psi(\Phi(a))(g, x) = (\phi(a)(g))(x) = a(g, x).$$

Step 4. $\Phi(\Psi(\theta)) = \theta$

For $g \in G$ and $x \in X$,

$$\Phi(\Psi(\theta))(g)(x) = \Psi(\theta)(g, x) = \theta(g)(x).$$

Hence, $\Phi(\Psi(\theta))(g) = \theta(g)$ for all $g \in G$, which means $\Phi(\Psi(\theta)) = \theta$. This completes the proof of Step 4. $\square$

The group homomorphism θ_a is called the induced group homomorphism, and the action a_θ is called the induced action.

Using the same argument as in the proof of Proposition 2, we obtain a bijection between

$$\mathcal{A}_\mathcal{C}(G, X) := \{a : G \times X \rightarrow X \mid a \text{ is an action via } \mathcal{C}\text{-arrows}\}$$

and

$$\text{Hom}(G, \text{Aut}_{\mathcal{C}}(X)).$$

Let us think about more abstract group actions.

Left translation action. Suppose G is a group and H is a subgroup of G . Then G acts on G/H by left translations; that is, for every $g \in G$ and $g'H \in G/H$, we define

$$g \cdot (g'H) := (gg')H.$$

This defines an action of G on G/H .

PROOF. For every $g'H \in G/H$, we have

$$e \cdot (g'H) = (eg')H = g'H,$$

where e is the neutral element of G . For every $g_1, g_2 \in G$, and $gH \in G/H$, the following holds

$$g_1 \cdot (g_2 \cdot gH) = g_1 \cdot ((g_2g)H) = (g_1(g_2g))H = ((g_1g_2)g)H = (g_1g_2) \cdot (gH).$$

This completes the proof. $\square$

Conjugation action Suppose G is a group. Then G acts on itself by conjugation. That is, for every $g \in G$ and $x \in G$, we define $g \cdot x := gxg^{-1}$. This defines an action of G on itself.

PROOF. For every $g \in G$, $e \cdot g = ege^{-1} = g$. For every $g_1, g_2, x \in G$, we have

$$g_1 \cdot (g_2 \cdot x) = g_1 \cdot (g_2xg_2^{-1}) = g_1(g_2xg_2^{-1})g_1^{-1} = (g_1g_2)x(g_1g_2)^{-1} = (g_1g_2) \cdot x.$$

$\square$

An alternative approach is using the following result that you will expand on as part of your homework assignment.

LEMMA 3. Let $c : G \rightarrow \text{Aut}(G)$, $g \mapsto c_g$, where $c_g(x) := gxg^{-1}$. Then c is a group homomorphism.

PROOF. By Lemma ?? (previous lecture), we have proved that $c_g \in \text{Aut}(G)$. Hence, c is a well-defined function. Next, we show that c is a group homomorphism. For all $g_1, g_2, x \in G$,

$$\begin{aligned} c_{g_1} \circ c_{g_2}(x) &= c_{g_1}(g_2xg_2^{-1}) = g_1(g_2xg_2^{-1})g_1^{-1} \\ &= (g_1g_2)x(g_1g_2)^{-1} = c_{g_1g_2}(x). \end{aligned}$$

Therefore, $c_{g_1} \circ c_{g_2} = c_{g_1 g_2}$, which implies that

$$c(g_1) \circ c(g_2) = c(g_1 g_2).$$

This completes the proof. $\square$

Let $a : G \times G \rightarrow G$ be the action induced by the group homomorphism $c : G \rightarrow \text{Aut}(G)$. This means

$$g \cdot x := (c(g))(x) = gxg^{-1},$$

which means this induced action is the conjugation action.

Left translation action on the space of functions Suppose $G \curvearrowright X$. Then the following defines an action of G on the set $\text{Fun}(X, Y)$ of functions on X ,

$$(g \cdot f)(x) := f(g^{-1} \cdot x).$$

When X is finite and $Y = \mathbb{C}$, then $\text{Fun}(X, \mathbb{C})$ is viewed as a vector space with the inner product

$$\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x) \overline{f_2(x)},$$

and it is easy to see that

$$\langle g \cdot f_1, g \cdot f_2 \rangle = \langle f_1, f_2 \rangle,$$

which means $G \curvearrowright L^2(X)$ by unitary transformations.

Large simple groups cannot act non-trivially on small sets. An important technique for showing that a group G is not simple is to prove that it acts non-trivially on a *small set*.

Suppose $a : G \times X \rightarrow X$ is a group action, and let $\theta_a : G \rightarrow S_X$ be the induced group homomorphism. Then $\ker \theta_a$ is called the kernel of the action a . By the first isomorphism theorem,

$$G / \ker \theta_a \hookrightarrow S_X.$$

Thus, if $|G|$ does not divide S_X and θ_a is not trivial, then $\ker \theta_a$ is a non-trivial proper normal subgroup. In particular, G is not simple.

PROPOSITION 4. *The kernel of the left translation action $G \curvearrowright G/H$ is*

$$\text{core}_G(H) := \bigcap_{g \in G} gHg^{-1}.$$

If N is a normal subgroup of G and $N \subseteq H$, then $N \subseteq \text{core}_G(H)$. Hence, $\text{core}_G(H)$ is the largest normal subgroup of G contained in H .

The group $\text{core}_G(H)$ is called the *normal core* of H in G .

PROOF OF PROPOSITION 4. Note that

$$\begin{aligned} x \in \ker \theta_a &\iff \forall g \in G, x \cdot (gH) = gH \\ &\iff \forall g \in G, g^{-1}xg \in H \\ &\iff x \in \bigcap_{g \in G} gHg^{-1}. \end{aligned}$$

This shows that $\text{core}_G(H)$ is the kernel of the left-translation action $G \curvearrowright G/H$. In particular, $\text{core}_G(H)$ is a normal subgroup of G .

If $N \subseteq H$ is a normal subgroup of G , then for all $g \in G$,

$$gNg^{-1} \subseteq gHg^{-1} \quad \text{and so} \quad N \subseteq gHg^{-1}.$$

Therefore, $N \subseteq \bigcap_{g \in G} gHg^{-1}$. This completes the proof. $\square$

PROPOSITION 5. Suppose p is the smallest prime factor of the order of G . If H is a subgroup of G and $[G : H] = p$, then H is a normal subgroup of G .

PROOF. Consider the left translation action $G \curvearrowright G/H$. Let $\theta : G \rightarrow S_{G/H}$ be the induced group homomorphism. Note that $|S_{G/H}| = p!$, so

$$|\text{Im}(\theta)| \quad \text{divides} \quad \gcd(|G|, p!) = p.$$

Since the left translation action is non-trivial, we deduce that $|\text{Im}(\theta)| = p$. By the first isomorphism theorem and Proposition 4, we deduce that

$$[G : \text{core}_G(H)] = p = [G : H].$$

Since $\text{core}_G(H) \subseteq H$ and $[G : \text{core}_G(H)] = [G : H]$, it follows that $H = \text{core}_G(H)$. Hence, H is a normal subgroup of G . $\square$

What is the kernel of the conjugation action? Note that $g \in G$ is in the kernel of the conjugation action if and only if for all $x \in G$

$$g \cdot x = x \iff \forall x \in G, gxg^{-1} = x \iff \forall x \in G, gx = xg \iff g \in Z(G).$$

Hence, the kernel of the conjugation action is the *center* $Z(G)$ of G .

Having an action $G \curvearrowright X$, we may ask what points of X are G -similar to a given point $x \in X$. In other words, using the symmetries of X provided by the

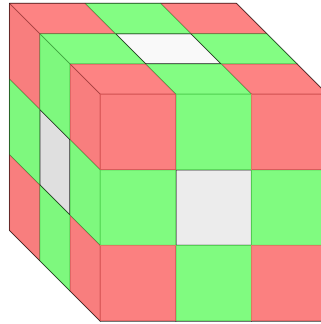
elements of G , which points of X can be reached starting from x ? This leads us to the definition of G -orbit of x .

DEFINITION 6. Suppose $G \curvearrowright X$. For $x \in X$, let

$$G \cdot x := \{g \cdot x \mid g \in G\}.$$

The set $G \cdot x$ is called the G -orbit of x .

Consider the group G of symmetries of a Rubik's Cube generated by the allowed moves. Then corner pieces can only be mapped to corner pieces, and edge pieces can only be mapped to edge pieces.



Here is another example. Let G be the group of symmetries of the following tiling of the plane. Then the G -orbit of P is shown in the figure below.

