1 Homework 9.

1. Suppose A is a unital commutative ring.

   (a) Suppose $\mathfrak{a}$ is an ideal of A, and let $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$, $\pi_{\mathfrak{a}}(x) := x + \mathfrak{a}$. Convince yourself that

   $$\pi_{\mathfrak{a}} : A[x] \to (A/\mathfrak{a})[x],\ \pi_{\mathfrak{a}}(\sum_{i=0}^{n} a_i x^i) := \sum_{i=0}^{n} \pi_{\mathfrak{a}}(a_i)x^i$$

   is a surjective ring homomorphism and its kernel is

   $$\mathfrak{a}[x] := \left\{ \sum_{i=0}^{n} a_i x^i \mid n \in \mathbb{Z}^+, a_i \in \mathfrak{a} \right\}.$$

   (b) Prove that if $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\mathfrak{p}[x] \in \mathrm{Spec}(A[x])$.

   (c) Prove that $\mathrm{Nil}(A[x]) = \mathrm{Nil}(A)[x]$.

   (d) Prove that

   $$A[x]^{\times} = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0 \in A^{\times}, a_1, \ldots, a_n \in \mathrm{Nil}(A), n \in \mathbb{Z}^+\}.$$

2. For every ring A, prove that

   $$A[x]/\langle x^2 - 2\rangle \simeq \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in A \right\}.$$

   (**Hint**. Send $f(x) \in A[x]$ to $f\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and show that the kernel is $\langle x^2 - 2\rangle$. Here is an alternative approach: for every $f(x) \in A[x]$, by long division, there exists unique $a, b \in A$ and $q(x) \in A[x]$ such that $f(x) = (x^2 - 2)q(x) + a + bx$. Deduce that $f(x) + \langle x^2 - 2\rangle = (a + bx) + \langle x^2 - 2\rangle$ for a unique pair of elements $a, b \in A$. Send $a + bx + \langle x^2 - 2\rangle$ to $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$.)

3. Prove that $\langle x, 2\rangle$ in $\mathbb{Z}[x]$ is not a principal ideal.

4. Let $\omega := \frac{-1+i\sqrt{3}}{2}$ and

   $$\mathbb{Z}[\omega] := \{a + b\omega \mid a, b \in \mathbb{Z}\}.$$

   Convince yourself that $\mathbb{Z}[\omega]$ is a subring of $\mathbb{C}$.

(a) Let $N(z) := |z|^2$ and check that

$$N(a + bw) = a^2 - ab + b^2$$

for every $a, b \in \mathbb{R}$.

(b) Prove that for every $z_1 \in \mathbb{Z}[\omega]$ and $z_2 \in \mathbb{Z}[\omega] \setminus \{0\}$ there are $q, r \in \mathbb{Z}[\omega]$ such that

$$z_1 = z_2 q + r \quad \text{and} \quad N(r) < N(z_2).$$

(c) Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain and so it is a PID.

(d) Prove that $\mathbb{Z}[\omega]^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$.

5. Suppose A is a unital commutative ring and $\mathfrak{a}$ is an ideal of A. Let

$$\sqrt{\mathfrak{a}} := \{a \in A \mid \exists n \in \mathbb{Z}^+, a^n \in \mathfrak{a}\}.$$

(a) Prove that $\sqrt{\mathfrak{a}}$ is an ideal of A and $\mathrm{Nil}(A/\mathfrak{a}) = \sqrt{\mathfrak{a}}/\mathfrak{a}$.

(b) Let $V(\mathfrak{a})$ be the set of *prime divisors* of $\mathfrak{a}$; that means

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Prove that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.