1 Homework 6.

1. For a group G, let $[G, G]$ be the group generated by $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$'s where $g_1, g_2 \in G$. This is called the *derived subgroup* of G.

   (a) Prove that $[G, G]$ is a characteristic subgroup.

   (b) Prove that for a normal subgroup N of G, G/N is abelian precisely when $[G, G] \subseteq N$.

   (c) Prove that $[S_n, S_n] = A_n$ for every integer $n \geq 3$.

2. Suppose $n \geq 5$ and $m \geq 2$ are integers.

   (a) Find the composition factors of S_n.

   (b) Prove that if N is a non-trivial proper normal subgroup of S_n, then $N = A_n$.

   (c) Find out for what values of m, S_m is solvable.

3. Suppose the following is a SES

$$1 \to G_1 \to G_2 \to G_3 \to 1.$$

   Prove that G_2 is solvable if and only if G_1 and G_3 are solvable.

4. Prove that there is no finite group G such that $[G, G] \simeq S_4$.

   (**Hint.** Suppose to the contrary that there exists a finite group G such that $[G, G] \simeq S_4$. Convince yourself that

$$P := \{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

   is the unique Sylow 2-subgroup of A_4. Deduce that P is a characteristic subgroup of A_4. Consider the action of G on $[G, G] \simeq S_4$ by via conjugation. Since A_4 and P are characteristic subgroups of S_4, obtain an action by automorphisms on A_4/P. This gives you a group homomorphism from G to $\mathrm{Aut}(A_4/P)$. Argue why this implies that $[G, G]$ acts trivially on A_4/P. This means S_4 acts trivially on A_4/P by conjugations. Observe that

$$(1\ 2)(1\ 2\ 3)(1\ 2)P \neq (1\ 2\ 3)P,$$

   and get a contradiction.)

5. Prove that $D_\infty := \{ax + b \mid a \in \{\pm 1\}, b \in \mathbb{Z}\}$ under composition is an infinite solvable group which is generated by two elements of order 2. Find the center $Z(D_\infty)$ of D_∞.

   (**Hint.** Think about the symmetries of the integer grid in the real line.)

6. Suppose G is a group. For all $x, y \in G$, let

$$[x, y] := xyx^{-1}y^{-1} \quad \text{and} \quad {}^x y := xyx^{-1}.$$

   Then Hall's equation asserts that

$$[[x, y],^y z][[y, z],^z x][[z, x],^x y] = 1$$

   for all $x, y, z \in G$. You can check this on your own and use it in this exercise. For two subgroups H_1 and H_2 of G, $[H_1, H_2]$ denotes the group generated by

$$\{[x_1, x_2] \mid x_1 \in H_1,\ x_2 \in H_2\}.$$

   Let $\gamma_1(G) := G$ and $\gamma_{k+1}(G) := [G, \gamma_k(G)]$ for all positive integers k – $\{\gamma_i(G)\}_i$ is called the *lower central series* of G.

   (a) Suppose H, K, L are normal subgroups of G. Prove that

$$[[H, K], L] \leq [[K, L], H][[L, H], K].$$

   (b) Prove that for every positive integers m and n,

$$[\gamma_m(G), \gamma_n(G)] \subseteq \gamma_{m+n}(G).$$

   (**Hint.** (1) Since H, K, L are normal subgroups,

$$[[K, L], H][[L, H], K]$$

   is a normal subgroup of G. Consider $\overline{G} := G/[[K, L], H][[L, H], K]$, let $\overline{H}$, $\overline{K}$, and $\overline{L}$ be the quotient of H, L, and K by $[[K, L], H][[L, H], K]$. Use Hall's equation, and obtain that for all $h \in \overline{H}$, $k \in \overline{K}$, and $l \in \overline{L}$, we have that $[h, k]$ and l commute. Deduce that $[\overline{H}, \overline{K}]$ commute with l. Obtain that $[[\overline{H}, \overline{K}], \overline{L}] = 1$.

   (2) Use induction on m and part (a).)