1 Homework 7.

1. Suppose $G$ is a group. For all $x, y \in G$, let
   \[ [x, y] := xyx^{-1}y^{-1} \quad \text{and} \quad x y := xyx^{-1}. \]

   Then Hall’s equation asserts that
   \[ [[[x, y], y], z][[y, z], x][[z, x], x]y = 1 \]
   for all $x, y, z \in G$. You can check this on your own and use it in this exercise.

   (a) Suppose $H, K, L$ are normal subgroups of $G$. Prove that
   \[ [[[H, K], L], H][[L, H], K] \leq [[[K, L], H][L, H], K]. \]

   (b) Prove that for every positive integers $m$ and $n$,
   \[ [\gamma_m(G), \gamma_n(G)] \subseteq \gamma_{m+n}(G). \]

   (Hint. (1) Since $H, K, L$ are normal subgroups,
   \[ [[[K, L], H][L, H], K] \]
   is a normal subgroup of $G$. Consider $\overline{G} := G/[[[K, L], H][L, H], K]$, let $\overline{H}$, $\overline{K}$, and $\overline{L}$ be the quotient of $H$, $L$, and $K$ by $[[[K, L], H][L, H], K]$. Use Hall’s equation, and obtain that for all $h \in \overline{H}$, $k \in \overline{K}$, and $l \in \overline{L}$, we have that $[h, k]$ and $l$ commute. Deduce that $[\overline{H}, \overline{K}]$ commute with $l$. Obtain that $[[\overline{H}, \overline{K}], \overline{L}] = 1$.

   (2) Use induction on $m$ and part (a). )

2. The Frattini subgroup $\Phi(G)$ of a group $G$ is the intersection of all of its maximal subgroups. Suppose $G$ is a finite group. Notice that under an automorphism of $G$, a maximal subgroup is sent to a maximal subgroup, and so the Frattini subgroup $\Phi(G)$ is a characteristic subgroup.

   (a) Suppose $H$ is a subgroup of $G$ and $H\Phi(G) = G$. Prove that $H = G$.

   (b) Suppose $S \subseteq G$. Prove that $\langle S \rangle = G$ if and only if $\langle \pi(S) \rangle = G/\Phi(G)$
   where $\pi : G \to G/\Phi(G)$ is the natural quotient map.
(c) Prove that $\Phi(G)$ is nilpotent.

(d) Prove that $G$ is nilpotent if and only if $G/\Phi(G)$ is nilpotent.

**(Hint.)** (1) Suppose to the contrary that $H$ is a proper subgroup, and let $M$ be a maximal subgroup of $G$ which contains $H$. Argue why $H\Phi(G) \subseteq M$.

(2) Suppose $P$ is a Sylow $p$-subgroup of $\Phi(G)$. Argue why $G = N_G(P)\Phi(G)$ (The Frattini argument). Deduce that $G = N_G(P)$.

(3) Suppose $Q$ is a Sylow $p$-subgroup of $G$. Assuming $G/\Phi(G)$ is nilpotent, deduce that $Q\Phi(G)$ is a normal subgroup of $G$. Use the Frattini argument and show that $N_G(Q)Q\Phi(G) = G$. Obtain that $N_G(Q) = G$.

3. Suppose $P$ is a finite group and $|P| = p^n$ where $p$ is prime and $n$ is a positive integer. Let $\text{Max}(P)$ be the set of all maximal subgroups of $P$.

(a) Prove that for all $M \in \text{Max}(P)$, $P/M \simeq \mathbb{Z}/p\mathbb{Z}$.

(b) Prove that $P/\Phi(P)$ is isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^{\text{Max}(P)}$.

(c) Prove that $\Phi(P) = P^p[P, P]$ where

$$P^p[P, P] := \{x^py \mid x \in P, y \in [P, P]\}.$$ 

(d) Suppose $P = \langle S \rangle$ and a proper subset of $S$ does not generate $P$. Prove that $|S| = \dim_{\mathbb{Z}/p\mathbb{Z}}(P/P^p[P, P])$.

**(Hint.)** (1) Since $P$ is nilpotent, $M$ is a proper subgroup of $N_G(M)$. Deduce that $M$ is a normal subgroup. Use the correspondence theorem and obtain that $G/M$ has no non-trivial subgroup. Deduce that $G/M \simeq \mathbb{Z}/p\mathbb{Z}$.

(2) Consider the group homomorphism,

$$\pi : P \rightarrow \prod_{M \in \text{Max}(P)} P/M, \quad \pi(x) := (xM)_{M \in \text{Max}(P)}$$

Argue why the kernel of $\pi$ is $\Phi(P)$. Deduce that $P/\Phi(P)$ is isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^{\text{Max}(P)}$.

(3) Use part (b) and deduce that $x^p \in \Phi(P)$ and $[x_1, x_2] \in \Phi(P)$ for all $x, x_1, x_2 \in P$. Obtain that $P^p[P, P] \subseteq \Phi(P)$.
(4) Notice that $P/[P,P]$ is an abelian group, and so $x \mapsto x^p$ is a group homomorphism from $P/[P,P]$ to itself. The image of this group homomorphism is $P^p/[P,P]$. Hence $P^p[P,P]$ is a normal subgroup of $P$. Consider

$$V := P/P^p[P,P].$$

Notice that $V$ is a finite abelian group and every non-trivial element has order $p$. Use the additive notation for $V$. Notice that $V$ can be viewed as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Then every non-zero element $v \in V$ is part of a $\mathbb{Z}/p\mathbb{Z}$-basis of $V$. Hence there is a subspace $W$ of codimension 1 which does not contain $v$. Notice that $V/W \cong \mathbb{Z}/p\mathbb{Z}$, and so $W$ is a maximal subgroup of $V$. Deduce that for all $x \not\in P^p[P,P]$ there is $M \in \text{Max}(P)$ such that $x \not\in M$. Hence $\Phi(P) \subseteq P^p[P,P].$

(5) Notice that $P = \langle S \rangle$ implies that $\pi(S)$ generates $P/\Phi(P)$. By part (c), $P/\Phi(P)$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$. Hence the $\mathbb{Z}/p\mathbb{Z}$-span of $\pi(S)$ is $P/\Phi(P)$. Therefore there is a $\mathbb{Z}/p\mathbb{Z}$-basis $S'$ of $P/\Phi(P)$ which is a subset of $\pi(S)$. Suppose $S' \subseteq S$ is such that $|S'| = |S'|$ and $\overline{S'} = \pi(S')$. Argue why $\langle S' \rangle = P$. Deduce that $S' = S$, and obtain that $|S| = \dim_{\mathbb{Z}/p\mathbb{Z}}(P/\Phi(P))$.

( Remark. Part (d) implies that every two minimal generating set of $P$ have the same cardinality. This statement is false if $P$ is not a finite $p$-group. For instance $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is a cyclic group and so it can be generated by 1 element, but the set $\{(1,0),(0,1)\}$, which has 2 elements, is also a minimal generating set of $G$. )