

1 Homework 4.

1. Suppose D_{2n} is a dihedral group. Prove that there exists a splitting SES of the form $1 \rightarrow C_n \rightarrow D_{2n} \rightarrow C_2 \rightarrow 1$ where C_k is the cyclic group of order k .
2. Suppose G is a group.

- (a) Show that if N_1 and N_2 are normal subgroups of G and $N_1 \cap N_2 = \{1\}$, then for all $x_1 \in N_1$ and $x_2 \in N_2$, $x_1 x_2 = x_2 x_1$.
- (b) Suppose N_1, \dots, N_k are normal subgroups of G and $N_i \cap N_j = \{1\}$ for all $i \neq j$. Prove that

$$f : \prod_{i=1}^k N_i \rightarrow N_1 \cdots N_k, \quad f(x_1, \dots, x_k) := x_1 \cdots x_k$$

is a group homomorphism.

- (c) Suppose N_1, \dots, N_k are normal subgroups of G , and for all i ,

$$N_i \cap N_1 \cdots N_{i-1} N_{i+1} \cdots N_k = \{1\}.$$

Prove that

$$f : \prod_{i=1}^k N_i \rightarrow N_1 \cdots N_k, \quad f(x_1, \dots, x_k) := x_1 \cdots x_k$$

is a group isomorphism.

3. Suppose in a finite group G for every proper subgroup H , $H \subsetneq N_G(H)$.
 - (a) Prove that all the Sylow subgroups of G are normal. Deduce that for all prime divisors of $|G|$, G has a unique Sylow p -subgroup.
 - (b) Prove that $G \simeq \prod_{p \text{ prime factor of } |G|} P_p$ where P_p is the unique Sylow p -subgroup of G .

(Hint. Use $N_G(N_G(P)) = N_G(P)$ for every Sylow subgroup P , and the previous problem.)

4. Suppose G is a finite group and A is a normal abelian subgroup of G . Let $s : G/A \rightarrow G$ be a *section* of the natural projection map; that means for all $h \in G/A$, we choose an element $s(h)$ from the coset h . Alternatively we can say that $s(h)A = h$. Notice that if s is a group homomorphism, then the standard SES $1 \rightarrow A \rightarrow G \rightarrow G/A \rightarrow 1$ splits. The goal of this exercise is to modify s and make it into a group homomorphism under suitable assumptions. Let $H := G/A$ and define the function

$$c : H \times H \rightarrow A, \quad c(h_1, h_2) := s(h_1)s(h_2)s(h_1h_2)^{-1}.$$

Notice that since $s(h_1h_2)A = h_1h_2 = s(h_1)As(h_2)A = s(h_1)s(h_2)A$, the image of c is indeed in A . Function c gives us an insight on how far s is from being a group homomorphism. Notice that since A is abelian, the conjugation action of G on A factors through an action of H . More precisely, for all $h \in H$ and $a \in A$, let

$$h \cdot a := s(h)as(h)^{-1},$$

and notice that this is a well-defined group action.

- (a) Prove that, for all $h_1, h_2, h_3 \in H$, we have

$$c(h_1, h_2)c(h_1h_2, h_3) = (h_1 \cdot c(h_2, h_3))c(h_1, h_2h_3).$$

(Since A is abelian, it is more customary to write this equation in an additive notation:

$$c(h_1, h_2) + c(h_1h_2, h_3) = h_1 \cdot c(h_2, h_3) + c(h_1, h_2h_3),$$

and this is called the 2-cycle relation.)

- (b) Prove that the standard SES $1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$ splits if and only if there exists a function $b : H \rightarrow A$ such that

$$c(h_1, h_2) = b(h_1)(h_1 \cdot b(h_2))b(h_1h_2)^{-1}.$$

(Again it is customary to write this equation in an additive notation:

$$c(h_1, h_2) = b(h_1) + h_1 \cdot b(h_2) - b(h_1h_2).$$

This is called a 2-boundary.)

- (c) In the above setting, assume that $\gcd(|A|, |H|) = 1$. Prove that every 2-cycle is a 2-boundary. Deduce that the standard SES

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$$

splits.

(Hint.

- (a) Since $s(h_1)s(h_2) = c(h_1, h_2)s(h_1h_2)$, we have

$$\begin{aligned} (s(h_1)s(h_2))s(h_3) &= c(h_1, h_2)s(h_1h_2)s(h_3) \\ &= c(h_1, h_2)c(h_1h_2, h_3)s(h_1h_2h_3). \end{aligned}$$

We also have

$$\begin{aligned} s(h_1)(s(h_2)s(h_3)) &= s(h_1)c(h_2, h_3)s(h_2h_3) = (h_1 \cdot c(h_2, h_3))s(h_1)s(h_2h_3) \\ &= (h_1 \cdot c(h_2, h_3))c(h_1, h_2h_3)s(h_1h_2h_3). \end{aligned}$$

- (b) Notice that this SES splits if and only if there exists a function $b : H \rightarrow A$ such that $\psi(h) := b(h)^{-1}s(h)$ is a group homomorphism. For all $h_1, h_2 \in H$, we have

$$\begin{aligned} \psi(h_1)\psi(h_2)\psi(h_1h_2)^{-1} &= b(h_1)^{-1}s(h_1)b(h_2)^{-1}s(h_2)s(h_1h_2)^{-1}b(h_1h_2) \\ &= b(h_1)^{-1}(h_1 \cdot b(h_2))^{-1}c(h_1, h_2)b(h_1h_2). \end{aligned}$$

Hence, the given SES splits precisely when

$$c(h_1, h_2) = (h_1 \cdot b(h_2))b(h_1)b(h_1h_2)^{-1}.$$

- (c) Use the additive notation for the abelian group A . Since $\gcd(|A|, |H|) = 1$, for every $a \in A$ there exists a unique $y \in A$ such that $|H|y = a$. Denote this element by $\frac{a}{|H|}$. Suppose c is a 2-cocycle and let

$$b : H \rightarrow A, \quad b(x) := \frac{\sum_{h \in H} c(x, h)}{|H|}.$$

Adding over the h_3 term in the 2-cocycle relation, deduce that

$$|H|c(h_1, h_2) + |H|b(h_1h_2) = |H|(h_1 \cdot b(h_2)) + |H|b(h_1),$$

and so c is a 2-coboundary.)

5. A subgroup K of G is called a *characteristic* subgroup if for all $\theta \in \text{Aut}(G)$, $\theta(K) = K$. Notice that every characteristic subgroup is normal.
- (a) Suppose N is a normal subgroup of G and K is a characteristic subgroup of N . Prove that K is a normal subgroup of G .
 - (b) We say a group H is *characteristically simple* if the only characteristic subgroups of H are 1 and H . Suppose N is a *minimal normal* subgroup of G ; that means if $M \leq N$ and $M \trianglelefteq G$, then either $M = \{1\}$ or $M = N$. Then N is characteristically simple.
6. Suppose G is a finite group.
- (a) Prove that a normal Sylow p -subgroup is a characteristic subgroup.
 - (b) Suppose H is a normal subgroup of G and $\gcd(|H|, [G : H]) = 1$. Prove that H is a characteristic subgroup.

(**Hint.** These parts are not related to each other. For the second part, suppose $\theta(H) \neq H$ for some $\theta \in \text{Aut}(G)$. show that $|\theta(H)H/H|$ divides $|\theta(H)|$ and $|G/H|$.)

7. In this problem, you will show that S_6 has an automorphism which is not an inner automorphism.
- (a) Show that S_5 has 6 Sylow 5-subgroups.
 - (b) Use the action of S_5 on $\text{Syl}_5(S_5)$ and show that S_6 has a subgroup H which is isomorphic to S_5 and for every $\sigma \in S_6$, $\text{Fix}(\sigma H \sigma^{-1}) = \emptyset$ where S_6 acts on $\{1, \dots, 6\}$.
 - (c) Consider the action $S_6 \curvearrowright S_6/H$ by left-translations. Argue that this action induces a group homomorphism $\theta : S_6 \rightarrow S_6$. Prove that $\text{Fix}(\theta(H)) \neq \emptyset$.
 - (d) Deduce that $\text{Aut}(S_6) \neq \text{Inn}(S_6)$.

(In this problem you are allowed to use the fact that if N is a normal subgroup of S_n , $[S_n : N] > 2$, and $n \geq 5$, then $N = \{1\}$.)

8. Prove that a group of order 36 is not simple.

(**Hint.** Suppose G is simple. Find the number of Sylow 3-subgroups of G . Consider the action of G on $\text{Syl}_3(G)$. Prove that the kernel of this action cannot be trivial.)