MATH200A, HOMEWORK ASSIGNMENT

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1. Special case of Burnside's theorem

Burnside used character theory of finite groups to prove that a group of order $p^n q^m$ is solvable. In this collection of problems you will prove that a group G of order $p^n q$ where p and q are primes is solvable.

(1) Prove that to show the above claim it is enough to argue why a group of order $p^n q$ is not a non-abelian simple group. (So in the rest of this problem you will assume to the contrary that G is a non-abelian simple group of order $p^n q$.)
(2) Prove that G has exactly q Sylow p-subgroups.
(3) Let Q be maximal among the intersections of pairs of Sylow p-subgroups of G. Suppose $Q \neq 1$ and let $H := N_G(Q)$. Prove that H has at least 2 Sylow p-subgroups.
(4) Prove that H has exactly q Sylow p-subgroups.
(5) Reprove problem 3 in your midterm to get a contradiction and deduce that the intersection of any two distinct Sylow p-subgroups of G is trivial.
(6) Prove that $G \setminus (\bigcup_{i=1}^{q}(P_i \setminus \{1\}))$ has exactly q elements, and deduce that G has a unique Sylow q-subgroup and get a contradiction.

2. Frattini subgroup

The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G. In this collection of problems you will investigate various properties of Frattini subgroups. All the groups in these problems are finite.

(1) Prove that $\Phi(G)$ is a characteristic subgroup.
(2) Suppose $f : G_1 \to G_2$ is an onto group homomorphism. Prove that $f(\Phi(G_1)) \subseteq \Phi(G_2)$.
(3) Suppose H is a subgroup of G. Prove that

$$H = G \text{ if and only if } H\Phi(G) = G.$$

(4) Let $\pi : G \to G/\Phi(G)$ be the natural projection map. Suppose $S \subseteq G$. Prove that

$$\langle S \rangle = G \text{ if and only if } \langle \pi(S) \rangle = \pi(G).$$

(5) Prove that $\langle S \rangle = G$ if and only if $\langle S \setminus \Phi(G) \rangle = G$.

(6) Let $d(G)$ be the minimum number of generators of G. Prove that

$$d(G) = d(G/\Phi(G)).$$

(7) Prove that $\Phi(G)$ is a nilpotent group (if G is a finite group). (Hint: Use Frattini's argument to show any Sylow subgroup of $\Phi(G)$ is normal.)

Next you will prove that $\Phi(G) = G^p[G,G]$ if G is a finite p-group. Notice that G^p is the set of the p-th powers of elements of G, and it is not necessarily a subgroup of G. It is, however, a subgroup if G is abelian. So $(G/[G,G])^p$ is a subgroup of $G/[G,G]$ and its preimage in G is $G^p[G,G]$. Therefore $G^p[G,G]$ is a subgroup of G.

(1) Prove that any maximal subgroup M of G is normal and G/M is a cyclic group of order p.
(2) Prove that $G^p[G,G] \subseteq \Phi(G)$.
(3) Suppose V is a finite dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$. Prove that $\Phi(V) = 0$.
(4) Suppose $\overline{G}$ is an abelian group and $g^p = 1$ for any $g \in \overline{G}$. For any $i \in \mathbb{Z}/p\mathbb{Z}$ and $g \in \overline{G}$, let $i \cdot g := g^i$. Convince yourself that $\cdot$ gives us a well-defined scalar product and we can view $\overline{G}$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Deduce that $\Phi(\overline{G}) = 1$ if $\overline{G}$ is a finite abelian and $g^p = 1$ for any $g \in \overline{G}$. (You do not need to write anything for this part.)
(5) Prove that $\Phi(G/(G^p[G,G])) = 1$.
(6) Prove that $\Phi(G) \subseteq G^p[G,G]$; and deduce that $\Phi(G) = G^p[G,G]$.
(7) Prove that $d(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} G/(G^p[G,G])$.
(8) Suppose S is a minimal generating set; that means $\langle S \rangle = G$ and $\langle S' \rangle \neq G$ for any proper subset S' of S. Prove that $|S| = d(G)$.
(9) Does the claim of the previous part hold for a finite group which is not a p-group?