Name: ______________________________

PID: ______________________________

Section: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Suppose G is a finite group, and P is a Sylow p-subgroup. Suppose $|\mathrm{Syl}_p(G)| \geq [G : P]$, where $\mathrm{Syl}_p(G)$ is the set of all the Sylow p-subgroups of G. Prove that for any $g \in G$ we have $g \in \langle P, gPg^{-1} \rangle$.

By Sylow's 2^{nd} theorem, $|\mathrm{Syl}_p(G)| = [G : N_G(P)]$.
Since $P \leq N_G(P)$, we have $[G:P] \geq [G : N_G(P)]$. $\Rightarrow [G:P] \geq |\mathrm{Syl}_p(G)|$. 1

By assumption we have $|\mathrm{Syl}_p(G)| \geq [G:P]$. 2

1, 2 imply that $|\mathrm{Syl}_p(G)| = [G:P]$. And so $[G:P] = [G:N_G(P)]$.

And so $|P| = |N_G(P)|$.
$P \subseteq N_G(P)$ $\Rightarrow P = N_G(P)$.

Let $H_g := \langle P \cup gPg^{-1} \rangle$. Then P and gPg^{-1} are Sylow p-subgps of H_g. ($|P| = |gPg^{-1}|$ is the largest power of p which divides $|G|$ and $|H_g| \mid |G|$; so $|P|$ is the largest power of p which divides $|H_g|$.)
Hence by Sylow's 2^{nd} theorem, $\exists h \in H_g$ s.t. $hPh^{-1} = gPg^{-1}$. And so $(h^{-1}g) P (h^{-1}g) = P$; this means $h^{-1}g \in N_G(P) = P$. Therefore

$g \in hP \subseteq H_g$.

2. Suppose G is a non-abelian finite group, $Z(G)$ is its center, and $G/Z(G)$ is a p-group.

   (a) (5 points) Prove that G has a unique Sylow p-subgroup P. (Hint: Think about $[G : N_G(P)Z(G)]$.)

$\forall g \in Z(G)$, $gPg^{-1} = P$. So $Z(G) \subseteq N_G(P)$; this implies $Z(G)\, N_G(P) = N_G(P)$. So $[G : N_G(P) Z(G)] = [G : N_G(P)] \mid [G:P]$.

Notice, since P is a Sylow p-subgp, $\gcd([G:P], p) = 1$. Therefore $\gcd([G : N_G(P)], p) = 1$. ① On the other hand,

$$\left.\begin{array}{l} [G : N_G(P)] = [G/Z(G) : N_G(P)/Z(G)] \mid |G/Z(G)| \\ |G/Z(G)| \text{ is a power of } p \end{array}\right\} \Rightarrow [G : N_G(P)] \text{ should be a power of } p \quad ②$$

①, ② imply that $[G : N_G(P)] = 1$; this implies $G = N_G(P)$; and so $P \triangleleft G$.

(b) (5 points) Prove that $p \mid |Z(G)|$.

<u>Solution 1.</u> $G \curvearrowright G$ by conjugation. $Z(G)$ is the kernel of this action. So we get an action $G/Z(G) \curvearrowright G$,

$$(gZ(G)) \cdot g' := g g' g^{-1}.$$

Since $G/Z(G)$ is a p-gp, we have

$$|G| \equiv |\text{fixed points of } G/Z(G)| \pmod p.$$

The set of fixed points of $G/Z(G) = \{g' \in G \mid \forall gZ(G),\ gZ(G) \cdot g' = g'\}$

$$= \{g' \in G \mid \forall g \in G,\ g g' g^{-1} = g'\} = Z(G).$$

Hence $|G| \equiv |Z(G)| \pmod p$. As $p \mid |G|$, we get that $p \mid |Z(G)|$.

<u>Solution 2.</u> Suppose to the contrary that $p \nmid |Z(G)|$. So $|G| = |G/Z(G)|\,|Z(G)|$ implies that $|G/Z(G)|$ is the largest power of p which divides $|G|$. And so $|P| = |G/Z(G)|$.

On the other hand, $P, Z(G) \triangleleft G$ and $\gcd(|P|, |Z(G)|) = 1$. Hence $P \cap Z(G) = 1$ as $|P \cap Z(G)| \mid |P|$ and $|P \cap Z(G)| \mid |Z(G)|$. And so $|P Z(G)| = |P|\,|Z(G)| = |G|$. Therefore $G = P Z(G)$ ①.

Since P is a finite p-gp, $Z(P) \neq 1$. Suppose $g_0 \in P \setminus \{1\}$. Then by ①, $\forall g \in G$, $\exists g_p \in P$ and $z \in Z(G)$ s.t. $g = g_p \cdot z$. And so

$$g_0 g = g_0 g_p z \underset{\uparrow}{=} g_p g_0 z \underset{\uparrow}{=} g_p z g_0 = g g_0;$$

$\boxed{g_0 \in Z(P)}$ $\boxed{z \in Z(G)}$

this implies $g_0 \in P \cap Z(G)$ which is a contradiction.

(This is an extremely easy special case of $|G| = p(p+1)$ problem that we discussed in class.)

3. Suppose G is a group of order 56. Let P_2 be a Sylow 2-subgroup of G, and P_7 be a Sylow 7-subgroup of G.
   (a) (5 points) Prove that either P_2 is normal in G or P_7 is normal in G.

Let $n_7 = |\mathrm{Syl}_7(G)|$. By Sylow theorems we know: $n_7 = [G : N_G(P_7)] \mid 8$ and $n_7 \equiv 1 \pmod 7$. Looking at the set $\{1, 2, 4, 8\}$ of positive divisors of 8, we see that the only possibilities for n_7 are 1 and 8. If $n_7 = 1$, then there is a unique Sylow 7-subgp; and so $P_7 \triangleleft G$. Suppose $n_7 = 8$; and $P_7^{(1)}, \dots, P_7^{(8)}$ are distinct Sylow 7-subgps. If $i \neq j$, then $|P_7^{(i)} \cap P_7^{(j)}| \mid 7$ and it is not 7. So If $i \neq j$, then $P_7^{(i)} \cap P_7^{(j)}$. We also notice that, $\forall g \in P_7^{(i)} \setminus \{1\}$, $o(g) = 7$ as $1 \neq |\langle g \rangle| \mid 7$. On the other hand, $o(g) = 7 \Rightarrow |\langle g \rangle| = 7 \Rightarrow \langle g \rangle$ is a Sylow 7-subgp of G ($7 \mid |G|$ and $7^2 \nmid |G|$). So $\langle g \rangle = P_7^{(i)}$ for some i. Hence $\{g \in G \mid o(g) \neq 7\} = G \setminus \left(\bigcup_{i=1}^{8} (P_7^{(i)} \setminus \{1\}) \right)$. Thus

$$|\{g \in G \mid o(g) \neq 7\}| = |G| - 8 \times 6 \quad \longleftarrow \boxed{P_7^{(i)} \setminus \{1\} \text{ are disjoint}}$$
$$= 56 - 48 = 8$$
$$= |P_2|. \quad ①$$

$$\left(\forall g \in P_2 \Rightarrow o(g) \mid 8 \Rightarrow o(g) \neq 7 \right) \Rightarrow P_2 \subseteq \{g \in G \mid o(g) \neq 7\}. \quad ②$$

①, ② imply that $P_2 = \{g \in G \mid o(g) \neq 7\}$. Since conjugation does not change order, we have that, $\forall g' \in G$, $g' \{g \in G \mid o(g) \neq 7\} g'^{-1} = \{g \in G \mid o(g) \neq 7\}$

And so $P_2 \triangleleft G$.

(b) (5 points) Show that there are at least two non-isomorphic non-abelian groups of order 56. (Hint: You are allowed to use without proof that $|\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})| = (7)(24)$.)

<u>Claim</u>. $\exists$ a non-trivial group homomorphism $c : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$:

<u>Pf 1</u> $|\mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})| = \varphi(7) = 6$ and $2 \mid 6$. So by Cauchy's theorem $\exists$ an element of order 2 in $\mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$.

<u>Pf 2</u>. $c(1+2\mathbb{Z})(x) := -x$.

So there are non-trivial group homomorphisms $c_1 : \mathbb{Z}/8\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$, $c_2 : \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$, $c_3 : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$. To get c_i's it is enough to look at the following composit. group hom.:

$$\mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{c} \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z}); \quad \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{c} \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z});$$

$$a + 8\mathbb{Z} \mapsto a + 2\mathbb{Z} \qquad (x, y) \mapsto y$$

$$(\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{c} \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})$$

$$(x_1, x_2, x_3) \mapsto x_1$$

Since c_i's are non-trivial, $(\mathbb{Z}/8\mathbb{Z}) \ltimes_{c_1} (\mathbb{Z}/7\mathbb{Z})$, $(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \ltimes_{c_2} (\mathbb{Z}/7\mathbb{Z})$, $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes_{c_3} (\mathbb{Z}/7\mathbb{Z})$ are all non-abelian (Notice that in $H \ltimes_f N$, $h n h^{-1} = f(n)$ $\forall h \in H, n \in N$. So, if f is non-trivial, H does NOT commute with N; and so $H \not\trianglelefteq H \ltimes_f N$.) And Sylow 2-subgps of these three groups are not isomorphic.

You could use $|\mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^3)| = |\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})| = (7)(24)$ and Cauchy's theorem to find a non-trivial group hom. $c : \mathbb{Z}/7\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^3)$; and consider $(\mathbb{Z}/7\mathbb{Z}) \ltimes_c (\mathbb{Z}/2\mathbb{Z})^3$. And notice that here there is a unique Sylow 2-subgp.

Good Luck!