# Lecture 26: Prime and maximal ideals

Sunday, December 3, 2017 10:42 PM

**Def.** Suppose $I$ is a proper ideal of a unital commutative ring $R$.

(1) $I$ is called **prime** if $ab \in I \Rightarrow a \in I$ or $b \in I$.

(2) $I$ is called **maximal** if it is maximal in the set of proper ideals (with respect to inclusion); that means

$$I \subsetneq J \text{ and } J \trianglelefteq R \Rightarrow J = R.$$

**Proposition** Suppose $I \trianglelefteq R$. Then

(1) $I$ is prime $\Leftrightarrow R/I$ is an integral domain.

(2) $I$ is maximal $\Leftrightarrow R/I$ is a field.

(3) $I$ is maximal $\Rightarrow I$ is prime; and

$I$ is prime and $|R/I| < \infty \Rightarrow I$ is maximal.

**Pf.** (1) ($\Rightarrow$) $(a+I)(b+I) = \bar{0} \Rightarrow ab \in I$

$\Rightarrow a \in I$ or $b \in I$

$\Rightarrow a + I = \bar{0}$ or $b + I = \bar{0}$.

$I$ is proper $\Rightarrow R/I$ is not the zero ring.

($\Leftarrow$) $ab \in I \Rightarrow (a+I)(b+I) = \bar{0}$

$\Rightarrow a + I = \bar{0}$ or $b + I = \bar{0}$

$\Rightarrow a \in I$ or $b \in I$.

$R/I$ is not the zero ring $\Rightarrow I$ is proper.

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(2) ($\Rightarrow$) $a+I \neq \bar{0} \Rightarrow a \notin I \Rightarrow \left.\begin{array}{l}\langle a\rangle + I \supsetneq I \\ I \text{ max.}\end{array}\right\} \Rightarrow R = \langle a\rangle + I$

$\Rightarrow \exists\, r \in R,\ b \in I,\ 1 = ar + b$

$\Rightarrow (a+I)(r+I) = 1 - b + I = 1 + I$

$\Rightarrow a + I \in (R/I)^{\times}$.

And $R/I$ is not the zero ring.

($\Leftarrow$) Since $R/I$ is not the zero ring, $I$ is a proper ideal.

Suppose $I \subsetneq J$ and $J \trianglelefteq R$; and let $a \in J \setminus I$. Then

$a + I \neq \bar{0}$ in $R/I$. Hence $\exists\, r \in R$ s.t. $(a+I)(r+I) = 1 + I$.

$\Rightarrow \exists\, b \in I$ s.t. $ar + b = 1 \Rightarrow$

$R = \langle a\rangle + I \subseteq J$. Hence $J = R$.

(3). $I$: max. $\Rightarrow R/I$ : field $\Rightarrow R/I$ : integral domain $\Rightarrow I$ : prime.

. $\left.\begin{array}{l} I: \text{prime} \\ |R/I| < \infty \end{array}\right\} \Rightarrow \left.\begin{array}{l} R/I: \text{integ. domain} \\ |R/I| < \infty \end{array}\right\} \Rightarrow R/I$ : field $\Rightarrow I$ : max.

$\blacksquare$

<u>Def.</u> $\mathrm{Spec}(A) := \{\mathfrak{p} \trianglelefteq A \mid \mathfrak{p} : \text{prime ideal}\}$ and

$\mathrm{Max}(A) := \{\mathfrak{m} \trianglelefteq A \mid \mathfrak{m} : \text{maximal ideal}\}$.

Is there any maximal ideal? The main tool of proving the existence of maximal elements is Zorn's lemma.

# Lecture 26: Partially ordered sets; chains; upper-bound; maximal; Zorn's lemma

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Def. (Partially Ordered Set ; Poset) A non-empty set $\Sigma$ with an ordering $\preccurlyeq$ ; that means

$a \preccurlyeq b$ and $b \preccurlyeq c \Rightarrow a \preccurlyeq c$

$a \preccurlyeq b$ and $b \preccurlyeq a \Rightarrow a = b$

(Chain) A non-empty subset $C$ of a poset $(\Sigma, \preccurlyeq)$ is called a chain if $\forall a, b \in C$ either $a \preccurlyeq b$ or $b \preccurlyeq a$.

(Upper-bound) Suppose $(\Sigma, \preccurlyeq)$ is a poset, and $X$ is a non-empty subset of $\Sigma$. Then $m \in \Sigma$ is called an upper bound of $X$ if $\forall a \in X$, $a \preccurlyeq m$.

(Maximal) Suppose $(\Sigma, \preccurlyeq)$ is a poset, and $X$ is a non-empty subset of $\Sigma$. Then $m \in X$ is called a maximal element if it is an upper bound of $X$; that means,

$$\forall a \in \Sigma,\ m \preccurlyeq a \Rightarrow a \notin X.$$

Zorn's lemma. Suppose $(\Sigma, \preccurlyeq)$ is a poset, and any chain of $\Sigma$ has an upper-bound. Then $\Sigma$ has a maximal element.

# Lecture 26: Existence of maximal ideals; getting primes

Monday, December 4, 2017 8:22 AM

<u>Def.</u> A subset $S$ of a ring $A$ is called <u>multiplicatively closed</u> if $1 \in S$, and $\forall s_1, s_2 \in S$, $s_1 s_2 \in S$.

<u>Ex.</u> (1) $\forall f \in A$, $S_f := \{1, f, f^2, \ldots\}$

(2) $\forall \mathfrak{p}$: prime ideal of $A$, $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$

. $s_1, s_2 \in S_{\mathfrak{p}} \Rightarrow s_1 \notin \mathfrak{p}$ and $s_2 \notin \mathfrak{p}$

$\Rightarrow s_1 s_2 \notin \mathfrak{p} \Rightarrow s_1 s_2 \in S_{\mathfrak{p}}$.

. $\mathfrak{p}$ is proper $\Rightarrow 1 \notin \mathfrak{p} \Rightarrow 1 \in S_{\mathfrak{p}}$.

<u>Theorem.</u> $\forall \mathfrak{a} \trianglelefteq A$, $S \subseteq A$ multiplicatively closed

Suppose $\mathfrak{a} \cap S = \emptyset$, and let

$$\Sigma_{\mathfrak{a}, S} := \{\mathfrak{b} \trianglelefteq A \mid ① \ \mathfrak{a} \subseteq \mathfrak{b} \quad ② \ \mathfrak{b} \cap S = \emptyset\}.$$

Then (1) $\Sigma_{\mathfrak{a}, S}$ has a maximal element w.r.t. $\subseteq$.

(2) A maximal element $\mathfrak{p}$ of $\Sigma_{\mathfrak{a}, S}$ is a prime ideal.

<u>Cor. 1</u> Suppose $\mathfrak{a} \trianglelefteq A$, $S \subseteq A$ multiplicatively closed, and $S \cap \mathfrak{a} = \emptyset$. Then $\exists$ a prime ideal $\mathfrak{p}$ of $A$ s.t.

(1) $\mathfrak{a} \subseteq \mathfrak{p}$, (2) $\mathfrak{p} \cap S = \emptyset$.

# Lecture 26: Existence of maximal ideals; finding primes

Monday, December 4, 2017 8:35 AM

**Cor. 2.** Suppose $\mathfrak{a}$ is a proper ideal of $A$. Then there is a maximal ideal $\mathfrak{m}$ of $A$ s.t. $\mathfrak{a} \subseteq \mathfrak{m}$.

**Pf. of Cor. 2.** $S = \{1\}$ is multiplicatively closed. And $\mathfrak{b} \cap \{1\} = \emptyset \iff \mathfrak{b}$ is proper. So a maximal element $\mathfrak{m}$ of $\Sigma_{\mathfrak{a}, \{1\}} = \{\mathfrak{b} \trianglelefteq A \mid \mathfrak{a} \subseteq \mathfrak{b},\ \mathfrak{b} \cap \{1\} = \emptyset\}$ is a max. element of the set of all proper ideals; that means $\mathfrak{m}$ is a maximal ideal of $A$. ∎

We will prove the mentioned theorem in the next lecture.