Lecture 20: Free monoids

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Def. A set M with a binary operation $\cdot$ is called a monoid if

(1) (Associativity) $\forall x,y,z \in M$, $(x\cdot y)\cdot z = x\cdot(y\cdot z)$

(2) (Neutral element) $\exists e\in M$, $\forall x\in M$, $e\cdot x = x = x\cdot e$.

. Suppose X is a non-empty set. Let $\mathcal{L}(X)$ be the language in the alphabet of X; that means elements of $\mathcal{L}(X)$ (we call them words) are finite sequences with terms in X:

$$\omega = x_1 x_2 \cdots x_n \quad \text{where } x_i \in X.$$

We include the empty word $\varnothing$ in $\mathcal{L}(X)$.

Concatination defines a binary operator on $\mathcal{L}(X)$; that is

$$(x_1x_2\cdots x_n)\cdot(y_1y_2\cdots y_m) := x_1\cdots x_n y_1\cdots y_m.$$

Clearly $\cdot$ is an associative operator; and the empty word is the neutral element of $(\mathcal{L}(X),\cdot)$. So $(\mathcal{L}(X),\cdot)$ is a monoid. In fact $\mathcal{L}(X)$ is the free monoid generated by X; that means $\mathcal{L}(X)$ satisfies the following universal property:

(Universal Proprety of free objects.)

Any function f from X to a monoid M has a unique extension to a monoid homomorphism $\hat{f}: \mathcal{L}(X) \to M$.

The Universal Property of a free object is often described using the following diagram:

$$\begin{tikzcd} & \mathcal{L}(X) \arrow[d, dashed, "\exists!\, \hat{f}"] \\ X \arrow[ur, hook] \arrow[r, "f"'] & M \end{tikzcd}$$

Set Monoid

Remark. If monoid is changed to group, we get the definition of free group generated by X; if monoid is changed to k-algebra, we get the definition of free k-algebra; etc.

Pf of freeness of $\mathcal{L}(X)$.

Let $\hat{f}(x_1 \cdots x_n) := f(x_1) \cdot f(x_2) \cdot \ldots \cdot f(x_n)$ and $\hat{f}(\emptyset) = 1_M$; and check that $\hat{f}$ is a monoid homomorphism. Uniquess is clear! ■

Suppose $\{G_i\}_{i \in I}$ is a family of groups. Let X be the disjoint union of G_i's. (Notice that we can consider the set $G_i \times \{i\}$ instead of G_i, and think about it as a copy of G_i. This way we can make sure that G_i's are disjoint.)

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Let $\mathcal{L}(X)$ be the free monoid generated by X. For example

Suppose $G_1 = \mathbb{Z}/2\mathbb{Z}$ and $G_2 = \mathbb{Z}/3\mathbb{Z}$. First we pick isomorphic copies of G_1 and G_2 that are disjoint, say $G_1 = \{e, a\}$ and $a^2 = e$; $G_2 = \{1, b, b^2\}$ and $b^3 = 1$.

Then $e a 1 1 b b b^2 \in \mathcal{L}(X)$ and this is different from the word ab. The first word has length 7 and the 2nd word has length 2. To get a group structure we have to define an equivalency relation on $\mathcal{L}(X)$; let $\sim$ be the equi. relation generated by the following:

- $\omega_1 e \omega_2 \sim \omega_1 \omega_2$ if e is the neutral element of G_i for some $i \in I$.
- $\omega_1 x_1 x_2 \omega_2 \sim \omega_1 x_3 \omega_2$ if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$.

Let $\mathcal{F}(X) := \mathcal{L}(X)/\sim$.

<u>Claim.</u> $\omega_1 \sim \omega_1'$ and $\omega_2 \sim \omega_2' \Rightarrow \omega_1 \omega_2 \sim \omega_1' \omega_2'$

(try to convince yourself that this is true.)

Let $[\omega_1]_\sim \cdot [\omega_2]_\sim := [\omega_1 \omega_2]_\sim$. Then by the above claim

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This is a well-defined operator.

<u>Claim.</u> $(\mathcal{F}(X), \cdot)$ is a group.

<u>Pf.</u> • (Associative) $([\omega_1]\cdot[\omega_2])\cdot[\omega_3] = [\omega_1\omega_2]\cdot[\omega_3]$

$= [\omega_1\omega_2\omega_3]$

$[\omega_1]\cdot([\omega_2]\cdot[\omega_3]) = [\omega_1]\cdot[\omega_2\omega_3]$

$= [\omega_1\omega_2\omega_3]$

• (Neutral element) $[\omega]\cdot[\emptyset] = [\omega] = [\emptyset]\cdot[\omega]$

• (Inverse) $[x_1x_2\cdots x_n]\cdot[x_n^{-1}x_{n-1}^{-1}\cdots x_1^{-1}]$

$= [x_1x_2\cdots x_n x_n^{-1} x_{n-1}^{-1}\cdots x_1^{-1}]$.

$$x_1\cdots x_{n-1}\underbrace{x_n x_n^{-1}}\, x_{n-1}^{-1}\cdots x_1^{-1} \sim x_1\cdots x_{n-1}\,e\,x_{n-1}^{-1}\cdots x_1^{-1}$$

$$\sim x_1\cdots x_{n-1}x_{n-1}^{-1}\cdots x_1^{-1}$$

So by induction on n, we have

$$x_1\cdots x_n x_n^{-1}\cdots x_1^{-1} \sim \emptyset.$$

Similarly $[x_n^{-1}\cdots x_1^{-1}]\cdot[x_1\cdots x_n] = [\emptyset]$. ∎

$\mathcal{F}(X)$ is called the free product of G_i's; and it is denoted by $\underset{i\in I}{*} G_i$.

The universal property of free product of groups.

(<u>Warning</u>. In category theory, this is called the coproduct of these objects.)

Suppose G is a group and $f_i : G_i \to G$ are group homomorphisms

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Then there is a unique group homomorphism $\tilde{f}: \ast_{i\in I} G_i \to G$

such that $\tilde{f}|_{G_i} = f_i$. Alternatively

$$\mathrm{Hom}(\ast_{i\in I} G_i, G) \longrightarrow \prod_{i\in I} \mathrm{Hom}(G_i, G)$$
$$\tilde{f} \longmapsto (\tilde{f}|_{G_i})_{i\in I}$$

is a bijection.

<u>Pf.</u> Let X be the disjoint union of G_i's; and $\mathcal{L}(X)$ be the free monoid generated by X. Let $f: X \to G$,
$f(x) := f_i(x)$ if $x \in G_i$.

Since $\mathcal{L}(X)$ is the free monoid generated by X, there is a monoid homomorphism $\hat{f}: \mathcal{L}(X) \to G$ such that $\hat{f}|_X = f$. That means $\hat{f}|_{G_i} = f_i$ is a group homomorphism; and so

- $\hat{f}(e_{G_i}) = e_G$ where e_{G_i} is the neutral element of G_i and e_G is the neutral element of G.
- $\hat{f}(x_3) = \hat{f}(x_1)\hat{f}(x_2)$ if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$.

Next we show $\omega_1 \sim \omega_2 \Rightarrow \hat{f}(\omega_1) = \hat{f}(\omega_2)$.

Since $\sim$ is generated by the following relations, $\omega_1 e_{G_i} \omega_2 \sim \omega_1 \omega_2$

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and $\omega_1 x_1 x_2 \omega_2 \sim \omega_1 x_3 \omega_2$ if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$, it is enough to show

$$\hat{f}(\omega_1 e_{G_i} \omega_2) = \hat{f}(\omega_1 \omega_2) \quad (1) \quad \text{and} \quad \hat{f}(\omega_1 x_1 x_2 \omega_2) = \hat{f}(\omega_1 x_3 \omega_2) \quad (2)$$

if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$.

(1) $\hat{f}(\omega_1 e_{G_i} \omega_2) = \hat{f}(\omega_1) \hat{f}(e_{G_i}) \hat{f}(\omega_2)$ $\quad$ $\hat{f}$ is a monoid homomorphism

$= \hat{f}(\omega_1)\, e_G\, \hat{f}(\omega_2)$ $\quad$ $\hat{f}|_{G_i}$ is a gp hom.

$= \hat{f}(\omega_1) \hat{f}(\omega_2)$

$= \hat{f}(\omega_1 \omega_2)$ $\quad$ $\hat{f}$ is a monoid homo.

(2) $\hat{f}(\omega_1 x_1 x_2 \omega_2) = \hat{f}(\omega_1) \hat{f}(x_1) \hat{f}(x_2) \hat{f}(\omega_2)$ $\quad$ monoid hom.

$= \hat{f}(\omega_1) \hat{f}(x_3) \hat{f}(\omega_2)$ $\quad$ $\hat{f}|_{G_i} \in \mathrm{Hom}(G_i, G)$

$= \hat{f}(\omega_1 x_3 \omega_2)$ $\quad$ monoid hom.

Let $\tilde{f}([\omega]) := \hat{f}(\omega)$. The previous claim shows that $\tilde{f}$ is well-defined.

<u>Claim.</u> $\tilde{f} \in \mathrm{Hom}(\underset{i \in I}{*} G_i, G)$.

<u>Pf.</u> $\tilde{f}([\omega_1][\omega_2]) = \tilde{f}([\omega_1 \omega_2]) = \hat{f}(\omega_1 \omega_2) = \hat{f}(\omega_1) \hat{f}(\omega_2)$

$= \tilde{f}([\omega_1]) \tilde{f}([\omega_2])$

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. $\tilde{f}([\emptyset]) = \hat{f}(\emptyset) = e_G$

. $\tilde{f}([x_1 \cdots x_n]^{-1}) = \tilde{f}([x_n^{-1} \cdots x_1^{-1}]) = \hat{f}(x_n^{-1} \cdots x_1^{-1})$

$= \hat{f}(x_n^{-1})\, \hat{f}(x_{n-1}^{-1}) \cdots \hat{f}(x_1^{-1})$ monoid hom.

$= \hat{f}(x_n)^{-1}\, \hat{f}(x_{n-1})^{-1} \cdots \hat{f}(x_1)^{-1}$ $\hat{f}|_{G_i} \in \mathrm{Hom}(G_i, G)$

$= (\hat{f}(x_1)\, \hat{f}(x_2) \cdots \hat{f}(x_n))^{-1}$

$= \hat{f}(x_1 x_2 \cdots x_n)^{-1}$.

$= \tilde{f}([x_1 \cdots x_n])^{-1}$. ∎

<u>Claim</u>. $\tilde{f}|_{G_i} = f_i$.

<u>Pf</u>. $\forall\, x \in G_i$, $\tilde{f}([x]) = \hat{f}(x) = f_i(x)$. ∎

We have proved the existence of $\tilde{f} \in \mathrm{Hom}(\ast_{i \in I} G_i, G)$ s.t. $\tilde{f}|_{G_i} = f_i$. The uniqueness is clear as $\ast_{i\in I} G_i$ is generated by $X = \bigcup_{i \in I} G_i$. ∎

<u>Def</u>. For any non-empty set X, the free group generated by X is denoted by $F(X)$ and it is the free product of $|X|$ copies of $\mathbb{Z}$; $F(X) = \ast_{i \in I} \mathbb{Z}$.