Lecture 17: Finite solvable groups

Friday, November 3, 2017 11:41 AM

In the previous lecture we defined solvable group:

$$G^{(k)}=1 \text{ for some } k\in\mathbb{Z}^{\geq 0} \text{ or equivalently}$$

$$\exists\, 1=N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_0=G \text{ s.t. } N_i/N_{i+1} \text{ is abelian.}$$

We also mentioned

$$G: \text{solvable and simple} \Leftrightarrow G \text{ is cyclic gp of prime order}$$

Now we can prove:

Theorem. A finite group is solvable if and only if all of its composition factors are cyclic groups of prime order.

Pf. ($\Rightarrow$) Let $1=N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_0=G$ be a composition series. Since G is solvable, $\forall i$, N_i is solvable. And so $\forall i$, N_i/N_{i+1} is solvable and simple. Hence N_i/N_{i+1} is a cyclic group of prime order.

($\Leftarrow$) Let $1=N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_0=G$ be a composition series. By assumption, $\forall i$, N_i/N_{i+1} is a cyclic group. So by a theorem that we proved earlier, G is solvable. ∎

Lecture 17: Lower and upper central series

Thursday, November 2, 2017 11:16 PM

Def. Let $\gamma_1(G) := G$, $\gamma_{i+1}(G) := [G, \gamma_i(G)]$.

$\{\gamma_i(G)\}_{i=1}^{\infty}$ is called the <u>lower central series</u> of G.

Lemma (1) $\gamma_i(G) \lhd G$.

(2) $\gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots$.

(3) $Z\left(G/\gamma_{i+1}(G)\right) \supseteq \gamma_i(G)/\gamma_{i+1}(G)$.

Pf. (1) By induction. (2) $\left.\begin{array}{l}\gamma_i(G) \lhd G \\ G \lhd G\end{array}\right\} \Rightarrow \gamma_{i+1}(G) = [\gamma_i(G), G] \subseteq \gamma_i(G)$.

(3) $\forall\, g \in G$, $g_i \in \gamma_i(G)$, $[g, g_i] \in \gamma_{i+1}(G)$

$$\Rightarrow (g\,\gamma_{i+1}(G))(g_i\,\gamma_{i+1}(G)) = (g_i\,\gamma_{i+1}(G))(g\,\gamma_{i+1}(G)). \blacksquare$$

Def. Let $Z_0(G) = \{1\}$ and $Z(G/Z_i(G)) =: Z_{i+1}(G)/Z_i(G)$.

$\{Z_i(G)\}_{i=0}^{\infty}$ is called the <u>upper central series</u>.

Lemma. (1) $Z_i(G) \lhd G$

(2) $\{1\} = Z_0(G) \lhd Z_1(G) \lhd \cdots$

and $Z_i(G)/Z_{i-1}(G)$ is abelian.

Pf. (1) By induction on i and the fact that $Z(G/Z_i(G)) \lhd G/Z_i(G)$ claim follows.

(2) $Z_i(G)/Z_{i-1}(G) = Z(G/Z_i(G))$ and so it is abelian. $\blacksquare$

Lecture 17: Lower and upper central series

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__Theorem.__ Suppose G is a group. Then for a non-negative integer c

$$\gamma_{1+c}(G)=\{1\} \iff Z_c(G)=G.$$

__Pf.__ ($\Rightarrow$) we prove by induction on i that

$$\gamma_{1+c-i}(G)\subseteq Z_i(G).$$

- $\gamma_{1+c}(G)=\{1\}=Z_0(G)$ ✓
- $\gamma_{1+c-i}(G)\subseteq Z_i(G) \Rightarrow [G,\gamma_{c-i}(G)]\subseteq Z_i(G)$

  Let $\pi: G\to G/Z_i(G)$, $g\mapsto gZ_i(G)$. Then

  $[\pi(G),\pi(\gamma_{c-i}(G))]=\{\bar{1}\} \Rightarrow \pi(\gamma_{c-i}(G))\subseteq Z(\pi(G))$

$\Rightarrow \pi(\gamma_{c-i}(G))\subseteq Z(G/Z_i(G))=Z_{i+1}(G)/Z_i(G)$

$\Rightarrow \gamma_{c-i}(G)\subseteq Z_{i+1}(G).$

- So $\gamma_{1+c-c}(G)=G\subseteq Z_c(G)$; and the claim follows.

($\Leftarrow$) We prove by induction on i that $\gamma_i(G)\subseteq Z_{c-i+1}(G)$.

- $\gamma_1(G)=G\subseteq Z_c(G)$ ✓
- $\gamma_i(G)\subseteq Z_{c-i+1}(G) \Rightarrow \gamma_i(G)Z_{c-i}(G)/Z_{c-i}(G)\subseteq Z(G/Z_{c-i}(G))$

$\Rightarrow [G,\gamma_i(G)]\subseteq Z_{c-i}(G)$ (why?) $\Rightarrow \gamma_{i+1}(G)\subseteq Z_{c-i}(G)$. ✓

- Hence $\gamma_{c+1}(G)\subseteq Z_{(c+1)-(c+1)}(G)=Z_0(G)=\{1\}$. ∎

Lecture 17: Nilpotent groups

Friday, November 3, 2017 8:44 AM

Def. A group is called **nilpotent** if $\exists c \in \mathbb{Z}^{\geq 0}$ s.t.

$$\gamma_{1+c}(G) = \{1\}$$

(alternatively $Z_c(G) = G$.)

Proposition. Suppose G is a nilpotent group and $H \lneq G$. Then $N_G(H) \gneq H$.

Pf. Since G is nilpotent, $\exists c \in \mathbb{Z}^{\geq 0}$ s.t. $Z_c(G) = G$.

Since $H \neq G$, $\exists\ i_0 < c$ s.t. $Z_{i_0}(G) \subseteq H$ and $Z_{i_0+1}(G) \not\subseteq H$.

$\exists\ g \in Z_{i_0+1}(G) \setminus H$. Then, $\forall h \in H$, $(gZ_{i_0}(G))(hZ_{i_0}(G))$
$= (hZ_{i_0}(G))(gZ_{i_0}(G))$

$\Rightarrow\ h^{-1}g^{-1}hg \in Z_{i_0}(G) \subseteq H$

$\Rightarrow\ g^{-1}hg \in H$. And so $g^{-1}Hg \subseteq H$. (I)

Similarly $gHg^{-1} \subseteq H$. (II)

(I),(II) $\Rightarrow\ gHg^{-1} = H$; this implies $g \in N_G(H) \setminus H$. ∎