Lecture 16: A subgroup generated by a subset

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In the previous lecture, we saw that all the composition factors of a finite abelian group are cyclic groups of prime order.

$\boxed{Q}$ What can we say about a group if all of its composition factors are cyclic groups of prime order?

Let's recall an important def.

<u>Def</u>. Suppose G is a group and X is a non-empty subset of G. Then the subgroup generated by X is the smallest subgroup of G which contains X; and it is denoted by $\langle X\rangle$.

<u>Lemma</u>. $\langle X\rangle =$ the intersection of all the subgps of G containing X

(Exercise.)

<u>Lemma</u>. If $\theta: G\to H$ is a group homomorphism and $\emptyset\neq X\subseteq G$, then $\langle\theta(X)\rangle=\theta(\langle X\rangle)$.

<u>Pf</u>. • $\theta(\langle X\rangle)$ is a subgp of H which contains $\theta(X)$. So

$\langle\theta(X)\rangle\subseteq\theta(\langle X\rangle)$.

• $\theta^{-1}(\langle\theta(X)\rangle)$ is a subgp of G which contains X.

Lecture 16: Commutator of two subgroups

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So $\langle X\rangle \subseteq \theta^{-1}(\langle \theta(X)\rangle)$. Therefore $\theta(\langle X\rangle) \subseteq \langle \theta(X)\rangle$;

and the claim follows. $\blacksquare$

<u>Def.</u> • Commutator of $h,k\in G$ is $[h,k]=h^{-1}k^{-1}hk$.

• Commutator of two subgps H,K of G is

$$[H,K]=\langle \{[h,k] \mid h\in H, k\in K\}\rangle.$$

<u>Notice</u>. $\{[h,k] \mid h\in H, k\in K\}$ is not necessarily a group. It is important that $[H,K]$ is the group <u>generated</u> by the above set.

<u>Lemma</u>. Suppose $\theta: G\to H$ is a group homomorphism. Then

$$\forall g_1,g_2\in G,\quad [\theta(g_1),\theta(g_2)]=\theta([g_1,g_2]).$$

<u>Pf.</u> $\theta([g_1,g_2])=\theta(g_1^{-1}g_2^{-1}g_1g_2)=\theta(g_1)^{-1}\theta(g_2)^{-1}\theta(g_1)\,\theta(g_2)$

$$=[\theta(g_1),\theta(g_2)]. \quad \blacksquare$$

<u>Lemma</u>. Suppose $H,K\trianglelefteq G$. Then $[H,K]\trianglelefteq G$ and $[H,K]\subseteq H\cap K$.

<u>Pf.</u> $\forall g\in G$, $c_g: G\to G$, $c_g(g')=gg'g^{-1}$ is a group automorphism

$\Rightarrow c_g(\overbrace{\{[h,k] \mid h\in H, k\in K\}}^{X}) = \{[c_g(h),c_g(k)] \mid h\in H, k\in K\}$

$= \{[h',k'] \mid h'\in c_g(H), k'\in c_g(K)\} = \{[h',k'] \mid h'\in H, k'\in K\}$.

$\Rightarrow c_g(\langle X\rangle)=\langle c_g(X)\rangle=\langle X\rangle \Rightarrow c_g([H,K])=[H,K]$.

Lecture 16: Derived series

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$\Rightarrow [H,K] \triangleleft G.$

$[h,k] = \underbrace{(h^{-1}k^{-1}h)}_{\in K}\, \underbrace{k}_{\in K} \in K$, $[h,k] = \underbrace{h^{-1}}_{\in H}\underbrace{(k^{-1}hk)}_{\in H} \in H$

$\Rightarrow \{[h,k] \mid h\in H, k\in K\} \subseteq \underbrace{H\cap K}_{\text{a subgp}}$

$\Rightarrow [H,K] \subseteq H\cap K.$ ∎

<u>Def</u>. . The <u>derived subgroup</u> of G is $[G,G]$; and it is also denoted by $G^{(1)}$.

. The <u>derived series</u> of G is defined recursively:

$$G^{(0)} := G \quad , \quad G^{(i+1)} := [G^{(i)}, G^{(i)}] \quad \text{for any } i\in \mathbb{Z}^{\geq 0}.$$

<u>Lemma</u>. . Suppose $N \triangleleft G$. Then

G/N is abelian if and only if $[G,G] \subseteq N$.

<u>Pf</u>. Let $\pi: G \to G/N$, $\pi(g) = gN$. Then

$(\Rightarrow)$ $\forall g_1, g_2 \in G$, $\pi([g_1,g_2]) = [\pi(g_1), \pi(g_2)] = 1$ ← G/N is abelian

$\Rightarrow [g_1,g_2] \in \ker \pi = N \Rightarrow [G,G] \leq N.$

$(\Leftarrow)$ $\forall g_1, g_2 \in G$, $[g_1,g_2] \in [G,G] \subseteq N \rightarrow g_1^{-1}g_2^{-1}g_1g_2 \in N$

$\Rightarrow g_1g_2N = g_2g_1N \Rightarrow (g_1N)(g_2N) = (g_2N)(g_1N).$ ∎

Lecture 16: Solvable groups

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<u>Theorem.</u> Suppose G is a group. Then

$$G^{(k)}=\{1\} \iff \exists \ \{1\}=N_k \trianglelefteq N_{k-1} \trianglelefteq \cdots \trianglelefteq N_0 = G \text{ s.t. } N_i/N_{i+1} \text{ is abelian for any } i.$$

<u>Pf.</u> ($\Rightarrow$) Suppose $G^{(k)}=\{1\}$. Let $N_i := G^{(i)}$.

<u>Claim.</u> $G^{(i)} \triangleleft G$; in particular $G^{(i)} \triangleleft G^{(i-1)}$.

<u>Pf.</u> We prove this by induction on i.

- $G^{(0)} = G \triangleleft G$
- $G^{(i)} \triangleleft G \Rightarrow [G^{(i)}, G^{(i)}] \triangleleft G \Rightarrow G^{(i+1)} \triangleleft G$.

<u>Claim.</u> $G^{(i)}/G^{(i+1)}$ is abelian.

<u>Pf.</u> $G^{(i+1)} = [G^{(i)}, G^{(i)}]$; so by the previous lemma claim follows.

($\Leftarrow$) By induction, we show $G^{(i)} \subseteq N_i$; and so $G^{(k)}=1$.

- $G^{(0)} = G \subseteq N_0$.
- $G^{(i)} \subseteq N_i$

  N_i/N_{i+1} is abelian $\Rightarrow [N_i, N_i] \subseteq N_{i+1}$ $\Big\} \Rightarrow [G^{(i)}, G^{(i)}] \subseteq N_{i+1}$

  $\Rightarrow G^{(i+1)} \subseteq N_{i+1}$. ∎

<u>Def.</u> A group is called <u>solvable</u> if $\exists\, k \in \mathbb{Z}^+$ s.t. $G^{(k)}=\{1\}$.

<u>Lemma.</u> Suppose $\phi: G \to H$ is a group homomorphism. Then for any i, $\phi(G^{(i)}) = \phi(G)^{(i)}$.

<u>Pf.</u> (exercise).

Lecture 16: Solvable groups

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<u>Theorem</u>. Suppose G is a solvable group, $H \le G$, and $N \triangleleft G$. Then (1) H is solvable; (2) G/N is solvable.

<u>Pf</u>. (1) $\left.\begin{array}{l} H^{(i)} \subseteq G^{(i)} \quad \text{(by indu. on } i) \\ G^{(k)} = 1 \end{array}\right\} \Rightarrow H^{(k)} = 1.$

(2) $\pi : G \to G/N$ surjective group hom. $\Rightarrow \pi(G^{(k)}) = \pi(G)^{(k)}$

$\Rightarrow \pi(G)^{(k)} = \{\bar{1}\}$

$\Rightarrow (G/N)^{(k)} = \{\bar{1}\}$. ∎

<u>Lemma</u>. A solvable, simple gp is a cyclic gp of prime order.

<u>Pf</u>. $G^{(k)} = 1$ for some $k \in \mathbb{Z}^+$. Let k_0 be the smallest non-neg. integer s.t. $G^{(k_0)} = 1$. Notice, since G is simple, $G^{(0)} = G \neq \{1\}$. So $k_0 \ge 1$. Hence $1 \neq G^{(k_0-1)} \triangleleft G$

Since G is simple, $G^{(k_0-1)} = G$ and $G^{(k_0)} = G^{(1)} = 1$.

So G is abelian and simple. Hence G is a cyclic group of prime order (why?). ∎