Lecture 14: A_5 is simple

Monday, October 23, 2017 8:40 AM

<u>Lemma</u>. A_5 is simple.

<u>Pf</u>. Suppose $1 \neq N \trianglelefteq\!\!\!\!\!\neq\; A_5$.

<u>Case 1</u>. $3 \mid |N|$.

In this case, N has an element of order 3. If σ has order 3 and its cycle type is $p_1 \geq p_2 \geq \cdots \geq p_m$. Then $3 = \operatorname{lcm}(p_1, \ldots, p_m)$ and $p_1 + \cdots + p_m = 5$. So the only possib. is $3 \geq 1 \geq 1$. So N has a 3-cycle. Now using the previous lemma, we deduce that $N = A_5$.

<u>Case 2</u>. $5 \mid |N|$ and $3 \nmid |N|$.

So a Sylow 5-subgp P of A_5 is a subgp of N. As $N \trianglelefteq A_5$, all the Sylow 5-subgps are subgroups of N. But there are $4! = 24$ elements of order 5 in A_5. Hence $|N| \geq 25$ and $|N| \mid 60$. So $|N| = 30$, which contradicts our assumption that $3 \nmid |N|$.

<u>Case 3</u>. $\gcd(15, |N|) = 1$; and so $|N| \mid 4$.

Notice that $H := \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a Sylow 2-subgp of A_4; N should be a subgp of H. But $(1\ 5) H (1\ 5) = \{e, (5\ 2)(3\ 4), (5\ 3)(2\ 4), (5\ 4)(2\ 3)\}$; and $H \cap (1\ 5) H (1\ 5) = \{e\}$; which is a contradiction. ∎

Lecture 14: A comment on normal p-subgroups

Monday, October 30, 2017 11:09 AM

At the last step of the proof we used the following lemma which is of independent interest:

<u>Lemma</u>. Suppose N is a normal p-subgp of a finite group G. Then $N \subseteq \bigcap_{P \in Syl_p(G)} P$; and $\bigcap_{P \in Syl_p(G)} P$ is the normal core of a Sylow p-subgp.

<u>Pf.</u> Since N is a p-subgp of G, $\exists$ a Sylow p-subgp P_0 s.t. $N \subseteq P_0$. Hence, for any $g \in G$, $gNg^{-1} \subseteq gP_0g^{-1}$. As $N \trianglelefteq G$, we have $N \subseteq gP_0g^{-1}$ for any $g \in G$. Therefore

$$N \subseteq \bigcap_{g \in G} gP_0g^{-1} = \mathrm{cor}(P_0).$$

By Sylow's 2^{nd} theorem, $Syl_p(G) = \{gP_0g^{-1} \mid g \in G\}$. And so $\bigcap_{g \in G} gP_0g^{-1} = \bigcap_{P \in Syl_p(G)} P$. Therefore $N \subseteq \bigcap_{P \in Syl_p(G)} P$.

If $P_1 \in Syl_p(G)$, then $P_1 = gP_0g^{-1}$ for some $g \in G$. So

$$\mathrm{cor}(P_1) = \mathrm{cor}(gP_0g^{-1}) = g\,\mathrm{cor}(P_0)g^{-1} = \mathrm{cor}(P_0). \qquad \blacksquare$$

Lecture 14: Alternating group is simple

Wednesday, October 25, 2017 12:33 AM

<u>Theorem.</u> A_n is simple if $n \geq 5$.

<u>Pf.</u> We prove it by induction on n. We have already proved the base of induction.

Now suppose $\{e\} \neq N \triangleleft A_n$. Let $G_i := \{\sigma \in A_n \mid \sigma(i) = i\}$.

So, for any i, $G_i \simeq A_{n-1}$. Hence, for $n \geq 6$, G_i's are simple.

So either $N \cap G_i = G_i$ or $N \cap G_i = \{e\}$.

If for some i $N \cap G_i = G_i$, then N has a 3-cycle; and so $N = A_n$.

So w.l.o.g. we can and will assume $N \cap G_i = \{e\}$. So $\forall \sigma \neq \tau \in N$, $\forall i$, we have $\sigma(i) \neq \tau(i)$. $(*)$

Now suppose $e \neq \sigma \in N$ has cycle type $p_1 \geq p_2 \geq \cdots \geq p_m$.

<u>Case 1.</u> $p_1 \geq 3$.

$$\sigma = \underbrace{(a\ b\ c \cdots)}_{p_1} \underbrace{(\cdots)}_{p_2} \cdots \underbrace{(\ \ )}_{p_m} \in N$$

Suppose $\{e, f\} \cap \{a, b, c\} = \varnothing$; and consider

$\sigma' := (c\ e\ f)\ \sigma\ (c\ e\ f)^{-1} = (a\ b\ e \cdots)(\cdots)\cdots(\cdots) \in N$.

$\Rightarrow \sigma \neq \sigma'$ and $\sigma(a) = \sigma'(a)$; this contradicts $(*)$.

Lecture 14: Alternating group is simple

Wednesday, October 25, 2017 1:03 AM

Case 2. $p_1 < 3$.

So σ is a prod. of disj. transpositions:

$\sigma = (a\ b)(c\ d)\cdots(\cdots)$.

Suppose $\{e,f\} \cap \{a,b,c,d\} = \emptyset$. $(n \geq 6)$; and consider

$\sigma' := (c\ e\ f)\,\sigma\,(c\ e\ f)^{-1} = (a\ b)(e\ d)\cdots(\cdots)$

$\Rightarrow \sigma \neq \sigma'$ and $\sigma(a) = \sigma'(a)$; this contradicts $(*)$. ∎

Ex. Suppose G is a group of order $2m$ where m is an odd integer. Prove that G has a normal subgp of order m.

Pf. Since $2 \mid |G|$, by Cauchy's theorem $\exists g_0 \in G$ s.t. $o(g_0) = 2$. Now consider the action $G \curvearrowright G$ by left translations. Let $\phi: G \to S_G$ be the associated group homomorphism: $(\phi(g))(g') := gg'$. So we get a group homomorphism $\epsilon \circ \phi : G \to \{\pm 1\}$. By the 1^{st} isomorp. theorem $G/_{\ker \epsilon\circ\phi} \simeq \operatorname{Im}(\epsilon\circ\phi) \subseteq \{\pm 1\}$. So if we show $\phi(G)$ has an odd permutation, then $[G : \ker \epsilon\circ\phi] = 2$; this implies $N := \ker \epsilon\circ\phi$ is a normal subgp of order m.

Lecture 14: Applications of the sign function

Thursday, October 26, 2017 10:16 PM

<u>Claim</u>. $\phi(g_0)$ is an odd permutation.

<u>Pf.</u> Let's try to find the cycle type of $\phi(g_0)$.

① Since $g_0 \neq e$ and $\phi(g_0)(g) = g_0 g$, $\phi(g_0)$ has no fixed points.

② Since $g_0^2 = e$, $\phi(g_0)^2 = I_G$. And so $o(\phi(g_0)) = 2$.

① and ② imply the cycle type of $\phi(g_0)$ only consists of 2's.

Since $|G| = 2m$, the cycle type of $\phi(g_0)$ is $\underbrace{2 \leq \cdots \leq 2}_{m\text{-times}}$.

Hence $\phi(g_0)$ is a prod. of m transpositions.

Since m is odd, $\phi(g_0)$ is an odd permutation; and the claim follows. ■