Lecture 13: Sign function and transpositions

Monday, October 23, 2017 8:13 AM

In the previous lecture we defined $\Delta(x_1,\dots,x_n)=\prod_{i<j}(x_i-x_j)$ and $\Delta_\sigma(x_1,\dots,x_n)=\Delta(x_{\sigma(1)},\dots,x_{\sigma(n)})$. And proved that $\Delta_\sigma=\epsilon(\sigma)\,\Delta$.

<u>Theorem.</u> $\epsilon: S_n\to\{\pm1\}$ is a group homomorphism.

<u>Pf.</u> $\Delta_{\sigma\tau}=\epsilon(\sigma\tau)\,\Delta$ and

$$\begin{aligned}\Delta_{\sigma\tau}(x_i)&=\Delta(x_{\sigma\tau(i)})\\&=\Delta_\sigma(x_{\tau(i)})=\epsilon(\sigma)\,\Delta(x_{\tau(i)})\\&=\epsilon(\sigma)\,\epsilon(\tau)\,\Delta(x_i).\end{aligned}$$

So $\epsilon(\sigma\tau)=\epsilon(\sigma)\,\epsilon(\tau)$. $\blacksquare$

How can we determine $\epsilon(\sigma)$? In particular what is $\epsilon((a\ b))$?

A close look at the definition of $\epsilon(\sigma)$ shows us that

$$\epsilon(\sigma)=(-1)^{n_\sigma} \quad\text{where}\quad n_\sigma:=\left|\left\{(i,j)\ \middle|\ \begin{array}{l} i<j \text{ and} \\ \sigma(i)>\sigma(j)\end{array}\right\}\right|.$$

To understand n_σ better, we make an $n\times n$ matrix with i,j entry equals to $\mathrm{sgn}(\sigma(j)-\sigma(i))$. For instance for the identity element we get

$$\begin{bmatrix} 0 & + & \cdots & + \\ - & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & + \\ - & \cdots & - & 0 \end{bmatrix}.$$

Lecture 13: Sign of transpositions

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Let's see the matrix associated to $(1\ 2)$:

$$\begin{array}{cc} & \begin{array}{ccccc} 2 & 1 & 3 & \cdots & n \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 3 \\ \vdots \\ n \end{array} & \begin{bmatrix} 0 & - & + & \cdots & + \\ + & 0 & + & \cdots & + \\ - & - & 0 & & + \\ \vdots & \vdots & & \ddots & \\ - & - & & & 0 \end{bmatrix} \end{array}$$

As you can see $n_{(1\ 2)} = 1$.

(the number of $-$ in the upper-triang. part of the matrix.)

How about $(a\ b)$ where $a < b$?

$$\begin{array}{cc} & \begin{array}{ccccc} 1 \cdots (a-1) & b & (a+1) \cdots (b-1) & a & (b+1) \cdots n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ (a-1) \\ b \\ (a+1) \\ \vdots \\ (b-1) \\ a \\ (b+1) \\ \vdots \\ n \end{array} & \left[\begin{array}{c|c|c|c|c} \begin{matrix} 0 & & + \\ & \ddots & \\ - & & 0 \end{matrix} & \begin{matrix} + \\ \vdots \\ + \end{matrix} & + & \begin{matrix} + \\ \vdots \\ + \end{matrix} & + \\ \hline - \cdots - & 0 & - \cdots - & - & + \cdots + \\ \hline - & \begin{matrix} + \\ \vdots \\ + \end{matrix} & \begin{matrix} \cdot & & + \\ & \ddots & \\ - & & \cdot \end{matrix} & \begin{matrix} - \\ \vdots \\ - \end{matrix} & + \\ \hline - \cdots - & + & + \cdots + & 0 & + \cdots + \\ \hline - & \begin{matrix} - \\ \vdots \\ - \end{matrix} & - & \begin{matrix} - \\ \vdots \\ - \end{matrix} & \begin{matrix} \cdot & & + \\ & \ddots & \\ - & & \cdot \end{matrix} \end{array}\right] \end{array}$$

As you can see $n_{(a\ b)}$ is odd. So $\epsilon((a\ b)) = -1$.

The following theorem from theory of root systems gives us another way to think about n_σ:

Lecture 13: Parity

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Theorem. Let $s_1=(1\ 2)$, $s_2=(2\ 3)$, $\ldots$, $s_{n-1}=(n-1\ \ n)$. Then for any $\sigma\in S_n$,

$$\underbrace{|\{(i,j)\mid i<j,\ \sigma(i)>\sigma(j)\}|=\min\{m\mid \sigma=s_{i_1}\cdot s_{i_2}\cdot\ \cdots\ \cdot s_{i_m} \text{ for some choice of } i_1,\ldots,i_m\}}.$$

this is called the word length of σ with respect to $S=\{s_1,\ldots,s_{n-1}\}$.

Notice that $\underbrace{(b\ b-1)\cdots(a+2\ \ a+1)}_{\sigma}(a\ \ a+1)\underbrace{(a+2\ \ a+1)\cdots(b\ b-1)}_{\sigma^{-1}}$

$=(\sigma(a)\ \ \sigma(a+1))=(a\ b)$. So any permutation can be written as a product of elements of $\{s_1,\ldots,s_{n-1}\}$ (why?)

Theorem (1) Suppose $\sigma_1,\ldots,\sigma_n$ and $\tau_1,\ldots,\tau_m$ are transpositions.

If $\sigma_1\cdots\sigma_n=\tau_1\cdots\tau_m$, then $n\equiv m \pmod 2$.

(2) $\sigma\in\ker\epsilon \iff \sigma$ can be written as a product of even number of transpositions.

Pf ① $\epsilon(\sigma_1\cdots\sigma_n)=\epsilon(\tau_1\cdots\tau_m)\iff(-1)^n=(-1)^m\iff n\equiv m \pmod 2$.

② $\epsilon(\sigma_1\cdots\sigma_n)=1\iff 2\mid n$.

(and any σ can be written as a prod. of transpositions.) ∎

Def. $\ker\epsilon$ is called the alternating group, and it is denoted by A_n;. Elements of A_n are called <u>even</u>, and $\sigma\in S_n\setminus A_n$ is called odd.

Lecture 13: Even permutations and 15-puzzle

Friday, October 27, 2017 12:04 PM

In a 15-puzzle, you can rearrange numbers $1 \cdots 15$ in a 4×4 square by sliding the numbers to the empty spot.

$\boxed{Q}$ Can the arrangement in $\boxed{B}$ reached starting from $\boxed{A}$?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

A

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	

B

Solution. No! Any move is a transposition on $\{1, \ldots, 15, \square\}$; in fact any move is of the form $(\square \ i)$ for some $i \in [1 \cdots 15]$. Since at $\boxed{A}$ and $\boxed{B}$ the empty spot $\square$ is at the same place, the number of involved moves should be even: $\#$ of times $\square \uparrow = \#$ of times $\square \downarrow$ and

$\#$ of times $\square \leftarrow = \#$ of times $\square \rightarrow$.

So the final permutation should be an even permutation.

But $\boxed{B}$ is $(2 \ 1)$ which is odd. $\blacksquare$

Remark. In fact starting from $\boxed{A}$ one can reach to σ if and only if σ is an even permutation.

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$
$\sigma(5)$	$\sigma(6)$	$\sigma(7)$	$\sigma(8)$
$\sigma(9)$	$\sigma(10)$	$\sigma(11)$	$\sigma(12)$
$\sigma(13)$	$\sigma(14)$	$\sigma(15)$	

Lecture 13: 3-cycles and the alternating group

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Lemma. A_n is generated by 3-cycles if $n>2$.

Pf. Observations. $(a\ b)(b\ c)=(a\ b\ c)$

- $(a\ b)(c\ d)=(a\ b)(b\ c)(b\ c)(c\ d)$

$$=(a\ b\ c)(b\ c\ d)$$

So any even permutation is a product of 3-cycles; and the claim follows. $\blacksquare$

Lemma. Suppose $N \triangleleft A_n$, N contains a 3-cycle, and $n\geq 5$. Then $N=A_n$.

Pf. Step 1. If $n\geq 5$, then any two 3-cycles are conjugate in A_n.

Pf. Suppose τ_1 and τ_2 are 3-cycles. Then $\exists\, \sigma\in S_n$ s.t. $\sigma\tau_1\sigma^{-1}=\tau_2$. Since $n\geq 5$, $\exists\, a,b\in \{1,\ldots,n\}\setminus \operatorname{supp}\tau_1$. So $(a\ b)\tau_1=\tau_1(a\ b)$. Hence $\sigma(a\ b)\tau_1(a\ b)\sigma^{-1}=\tau_2$. Now either $\sigma\in A_n$ or $\sigma(a\ b)\in A_n$. In either case τ_1 is a conjugate of τ_2 in A_n.

Step 2. Since $N\triangleleft A_n$ and it contains a 3-cycle, by step 1 it contains all the 3-cycles. Therefore by the previous lemma $N=A_n$. $\blacksquare$