Lecture 12: Order of elements in a symmetric group

Tuesday, October 24, 2017 8:39 PM

In the previous lecture we proved that any $\sigma \in S_n$ can be written as a product of disjoint cycles, and this decomposition is unique up to reordering its factors.

Recall. Suppose G is a group, $g_1, g_2 \in G$ are torsion; that means $o(g_1), o(g_2) < \infty$. If $g_1 g_2 = g_2 g_1$, then

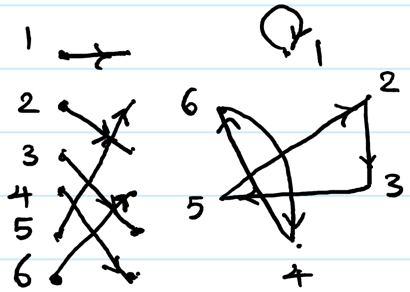
$$(*) \qquad o(g_1 g_2) = \mathrm{l.c.m.}(o(g_1), o(g_2)).$$

Corollary. Suppose the cycle decomposition of σ is given by $\tau_1 \cdots \tau_m$; and length of τ_i is l_i. Then

$$o(\sigma) = \mathrm{l.c.m.}(l_1, l_2, \ldots, l_m).$$

Pf. One can easily prove this using $(*)$, induction on m, and the fact that the order of a k-cycle is $k-1$. ∎

Def. The cycle type of a permutation $\sigma \in S_n$ is the partition of n given by the size of orbits $\langle \sigma \rangle \cdot i$.

Ex. 

$1+2+3$ is the cycle type of σ.

Lecture 12: Cycle type and conjugation

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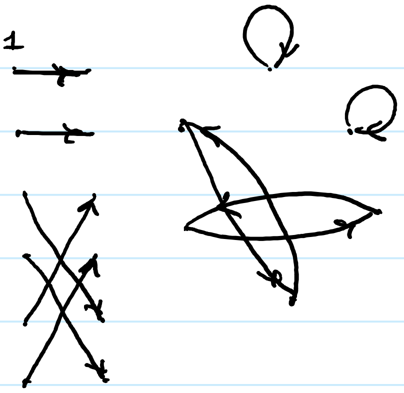
Ex. 

$1+1+2+2$ is the cycle type of σ.

Next we will see that two permutations are conjugate in S_n if and only if they have the same cycle type.

Lemma. $\sigma\underbrace{(i_1 \cdots i_k)}_{\tau}\sigma^{-1} = (\sigma(i_1) \cdots \sigma(i_k))$

Pf. $i \notin \{\sigma(i_1), \dots, \sigma(i_k)\} \Leftrightarrow \sigma^{-1}(i) \notin \{i_1, \dots, i_k\}$

$$\Leftrightarrow \tau(\sigma^{-1}(i)) = \sigma^{-1}(i)$$

$$\Leftrightarrow (\sigma\tau\sigma^{-1})(i) = i.$$

$$\cdot\ (\sigma\tau\sigma^{-1})(\sigma(i_j)) = \sigma\tau(i_j) = \begin{cases} \sigma(i_{j+1}) & \text{if } j \neq k \\ \sigma(i_1) & \text{if } j = k. \end{cases}$$

∎

Proposition $\sigma_1, \sigma_2 \in S_n$ are conjugate $\Leftrightarrow$ they have the same cycle type.

Pf. ($\Rightarrow$) Suppose $\tau_1 \cdots \tau_m$ is the cycle decomp. of σ_1. And $\sigma_2 = \sigma\sigma_1\sigma^{-1}$. Then $\sigma_2 = (\sigma\tau_1\sigma^{-1}) \cdots (\sigma\tau_m\sigma^{-1})$, and by the previous lemma, $\sigma\tau_i\sigma^{-1}$ is a cycle, it has the same length as τ_i, and $\operatorname{supp}(\sigma \tau_i \sigma^{-1}) = \sigma(\operatorname{supp} \tau_i)$.

Since σ is a bijection and $\operatorname{supp} \tau_i \cap \operatorname{supp} \tau_j = \emptyset$ for $i \neq j$, $\sigma(\operatorname{supp} \tau_i) \cap \sigma(\operatorname{supp} \tau_j) = \emptyset$. So $(\sigma \tau_1 \sigma^{-1}) \cdots (\sigma \tau_m \sigma^{-1})$ is the cycle decomposition of $\sigma_2 = \sigma \sigma_1 \sigma^{-1}$. Hence σ_1 and σ_2 are of the same cycle type.

$(\Leftarrow)$ Suppose σ_1 and σ_2 have the same cycle type; say $p_1 + \cdots + p_m = n$, $p_1 \geq p_2 \geq \cdots \geq p_m$ is their common cycle type. So

$$\sigma_1 = (\underbrace{\_\ \_\ \cdots\ \_}_{p_1})(\underbrace{\_\ \cdots\ \_}_{p_2}) \cdots (\underbrace{\_\ \_\ \cdots\ \_}_{p_m})$$

σ is a partition of $\{1, 2, \ldots, n\}$.

and
$$\sigma_2 = (\underbrace{\_\ \_\ \cdots\ \_}_{p_1})(\underbrace{\_\ \cdots\ \_}_{p_2}) \cdots (\underbrace{\_\ \_\ \cdots\ \_}_{p_m})$$

So the above σ is a bijection; that means $\sigma \in S_n$.

And by the previous lemma $\sigma \sigma_1 \sigma^{-1} = \sigma_2$; and the claim follows. ∎

Lecture 12: Transpositions

Monday, October 23, 2017 7:49 AM

<u>Def</u>. A 2-cycle $(a\ b)$ is called a transposition.

<u>Observation</u>. (Linking) Suppose $a_i \neq a_j$ for $i \neq j$. Then

$$(a_0\ a_1)(a_1\ a_2 \cdots a_m) = (a_0\ a_1 \cdots a_m)$$

- $(a_0\ a_1)(a_1\ a_2)\cdots(a_{m-1}\ a_m) = (a_0\ a_1 \cdots a_m)$

So any $\sigma \in S_n$ can be written as a product of transpositions.

Notice that, this is not a unique decomposition:

$$(1\ 2)(1\ 3)(1\ 2) = (2\ 3)$$

But the parity of the number of transpositions is the same in all of these decompositions.

Let $\Delta(x_1,\dots,x_n) := \prod_{1\le i<j\le n} (x_i - x_j)$; and

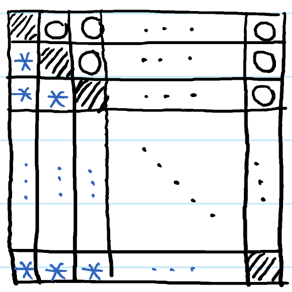
$$\Delta_\sigma(x_1,\dots,x_n) := \Delta(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \prod_{1\le i<j\le n} (x_{\sigma(i)} - x_{\sigma(j)})$$

for any $\sigma \in S_n$. Then

$$\prod_{i\neq j} (x_i - x_j) = (-1)^{\frac{n(n-1)}{2}} \Delta^2$$

$$\parallel$$

$$\prod_{i\neq j} (x_{\sigma(i)} - x_{\sigma(j)}) = (-1)^{\frac{n(n-1)}{2}} \Delta_\sigma^2$$



$\Rightarrow$ $\Delta_\sigma = \epsilon(\sigma)\,\Delta$ for some $\epsilon(\sigma) \in \{1,-1\}$.