Lecture 05: A general equation for group actions

Friday, October 6, 2017 8:34 AM

<u>Def.</u> let $X^G := \{x \in X \mid \forall g \in G,\ g\cdot x = x\}$. So

$$X^G = \bigcap_{g\in G} X^g .$$

(<u>Notice</u> that X^g, X^G are subsets of X;

G_x, kernel of the group action are subgroups of G.)

So far we know:

- ${}_G\backslash X = \{G\cdot x \mid x \in X\}$ is a partition of X.
- $G/G_x \to G\cdot x,\ gG_x \mapsto g\cdot x$ is a bijection.
- $|G\cdot x| = 1 \iff x \in X^G$.

Hence, if $|X| < \infty$, then

$$|X| = \sum_{G\cdot x \in {}_G\backslash X} |G\cdot x| = |X^G| + \sum_{\substack{|G\cdot x|>1 \\ G\cdot x \in {}_G\backslash X}} |G/G_x| .$$

- <u>$G \curvearrowright G$ by conjugation.</u>

The orbit of g is the set of conjugates of g;

it is called the conjugacy class of g, and in this course

we denote it by $Cl(g)$.

Lecture 05: Number of conjugates; the class equation

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The stabilizer of $g = \{g' \in G \mid g' g g'^{-1} = g\}$

$$= C_G(g)$$

is the centralizer of g.

So, if $|Cl(g)| < \infty$, then $|Cl(g)| = [G : C_G(g)]$.

The set "G^G" $:= \{g \in G \mid \forall g' \in G,\ g' g g'^{-1} = g\}$

is the center $Z(G)$ of G.

So, if G is a finite group, then

$$|G| = \sum_{Cl(g)} [G : C_G(g)] = |Z(G)| + \sum_{\substack{Cl(g) \\ g \notin Z(G)}} \frac{|G|}{|C_G(g)|}.$$

This is called the class equation.

Lecture 05: The kernel of an action and the normal core

Monday, October 9, 2017 10:40 AM

<u>Def.</u> The kernel of a group action $G \curvearrowright X$ is
$\{g \in G \mid \forall x \in X,\ g\cdot x = x\}$.

<u>Observe.</u> If $\psi: G \to S_X$ is the associated group homomorphism of a group action $G \curvearrowright X$, then the kernel of the group action is <u>$\ker(\psi)$</u>. And this is equal to $\bigcap_{x\in X} G_x$.

<u>Theorem.</u> Suppose H is a subgroup of G. Then the largest normal subgroup N of G which is a subgroup of H is $\bigcap_{g\in G} gHg^{-1}$ and it is called the <u>normal core of H</u>;
If $\mathrm{cor}(H) = \bigcap_{g\in G} gHg^{-1}$ and $[G:H]=n$, then

$$[G:\mathrm{cor}(H)] \le n!$$

<u>Pf.</u> Consider the action $G \curvearrowright G/H$ by the left translations; that means $g\cdot g'H := gg'H$. And let $\psi: G \to S_{G/H}$ be the associated group homomorphism. Then
$\forall gH \in G/H$, the stabilizer of $gH = \{g' \in G \mid g'gH = gH\}$

$$= \{g' \in G \mid g^{-1}g'g \in H\} = \{g' \in G \mid g' \in gHg^{-1}\} = gHg^{-1}.$$

So $\ker \psi = \bigcap_{g\in G} gHg^{-1}$.

Lecture 05: Normal core; A subgroup of index p

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So $\ker(\psi) = \mathrm{cor}(H)$; which implies $\mathrm{cor}(H)$ is a normal subgroup and $G/\mathrm{cor}(H) = G/\ker\psi \simeq \mathrm{Im}\,\psi \subseteq S_{G/H} \simeq S_n$.

Therefore $[G:\mathrm{cor}(H)] \leq n!$.

If $N \trianglelefteq G$ and $N \subseteq H$, then

$\forall g \in G$, $gNg^{-1} \subseteq gHg^{-1}$; so $N \subseteq gHg^{-1}$. Thus

$$N \subseteq \bigcap_{g\in G} gHg^{-1} = \mathrm{cor}(H);$$ and the claim follows. ∎

<u>Example</u> Suppose G is a finite group, and p is the smallest prime factor of $|G|$. Suppose $H \leq G$ and $[G:H] = p$. Prove that $H \trianglelefteq G$.

<u>Pf.</u> As we have seen in the proof of the previous theorem,

$G/\mathrm{cor}(H) \hookrightarrow S_p$; therefore by the Lagrange theorem $|G/\mathrm{cor}(H)| \mid p!$. On the other hand $\gcd(|G|, (p-1)!) = 1$.

So another application of the Lagrange theorem implies $\gcd(|G/\mathrm{cor}(H)|, (p-1)!) = 1$. Therefore $|G/\mathrm{cor}(H)| \mid p$.

Hence $|G/\mathrm{cor}(H)| = p = |G/H|$; this implies that $H = \mathrm{cor}(H)$; and so $H \trianglelefteq G$. ∎

Lecture 05: Normalizer

Sunday, October 8, 2017 5:27 PM

$G \curvearrowright \{H \mid H \leq G\}$ by conjugation.

- The orbit of H is the set of conjugates of H.
- The stabilizer of $H = \{g \in G \mid gHg^{-1} = H\}$

  It is denoted by $N_G(H)$ and it is called the normalizer of H in G.
- So # of conjugates of $H = [G : N_G(H)]$.
- Notice that $N_G(H)$ is the largest subgp of G which has H is a normal subgroup.